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Boundedness of the Discrete Riesz Transform on Discrete Morrey spaces

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Abstract. It is well known that the Riesz transform plays an important role in the theory of harmonic functions. This transform has been well studied on classical Lebesgue, Morrey, Sobolev, Besov, Campanato, etc. spaces. But its discrete version has not been well studied. In this paper, we prove that the discrete Riesz transform is a bounded operator in discrete Morrey spaces.

Key Words and Phrases: Riesz transform, discrete Riesz transform, Morrey spaces, discrete Morrey spaces, boundedness.
2010 Mathematics Subject Classifications: 44A15, 46B45, 42B35

1. Introduction

The j-th Riesz transform of the function $f \in L_p(\mathbb{R}^d)$, $1 \le p < +\infty$ is defined as the following singular integral:

$$R_j(f)(x) = \gamma_{(d)} \cdot \text{v.p.} \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)dy$$

$$= \gamma_{(d)} \cdot \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^d :\ |x-y|>\varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)dy,$$

where $\gamma_{(d)} = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$, $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$ is Euler's Gamma function.

It is well known (see [26]) that the Riesz transform plays an important role in the theory of harmonic functions. The boundary values of harmonically conjugate in the upper half space functions are interconnected by the Riesz transform. From the theory of singular integrals (see [6, 26]) it is known that the Riesz transform is a bounded operator in the space $L_p(\mathbb{R}^d)$, $1 < p < \infty$, that is, if $f \in L_p(\mathbb{R}^d)$, then $R_j(f) \in L_p(\mathbb{R}^d)$ and the inequality

$$\|R_j f\|_{L_p} \le C_p \|f\|_{L_p}$$

holds; in the case $f \in L_1(\mathbb{R}^d)$ only the weak inequality holds:

$$m\{x \in R^d :\ |(R_j f)(x)| > \lambda\} \le \frac{C_1}{\lambda} \|f\|_{L_1},$$

where m stands for the Lebesgue measure, C_p, C_1 are constants independent of f.

Denote by $l_p := l_p(Z^d)$, $p \geq 1$, the class of sequences $h = \{h_n\}_{n \in Z^d}$ satisfying the condition

$$\|h\|_{l_p} := \left(\sum_{n \in Z^d} |h_n|^p \right)^{1/p} < \infty,$$

where $Z^d := \{(n_1, ..., n_d) : \ n_k \in Z, \ k = 1, ..., d\}$ and Z is the set of integers.

Let $h = \{h_n\}_{n \in Z^d} \in l_p$, $p \geq 1$. Namely, the sequence $\widetilde{R}_j(h) = \{(\widetilde{R}_j h)_n\}_{n \in Z^d}$ is called the Riesz transform of the sequence h, where

$$(\widetilde{R}_j h)_n = \sum_{m \in Z^d, \ m \neq n} \frac{n_j - m_j}{|n - m|^{d+1}} \cdot h_m, \ n \in Z^d.$$

Note that if $h \in l_p$, $1 \leq p < \infty$, then from the Hölder inequality it follows that the series $\sum_{m \in Z^d, \ m \neq n} \frac{n_j - m_j}{|n-m|^{d+1}} \cdot h_m$ absolutely converges, and therefore the Riesz transform of the sequence h exists.

In [4, 5, 8, 9, 10, 17, 18, 20, 22] the boundedness of the Riesz transform in other function spaces was studied. But discrete version of the Riesz transform has not been well studied. In this paper, we prove that the discrete Riesz transform is a bounded operator on discrete Morrey spaces.

2. Discrete Morrey spaces

The classical Morrey spaces $M_{\lambda,p}$, $0 \leq \lambda \leq \frac{1}{p}$, $1 \leq p < \infty$ (see [1, 11, 19, 21, 23, 24, 25]), consist of the functions $f \in L_{p,loc}$ for which the following norm is finite

$$\|f\|_{M_{\lambda,p}} = \sup_{x \in R^d} \sup_{r > 0} \left[|B(x;r)|^{-\lambda} \|f\|_{L_p(B(x;r))} \right].$$

We note that if $\lambda = 0$, then $M_{\lambda,p} = L_p$; if $\lambda = \frac{1}{p}$, then $M_{\lambda,p} = L_\infty$ (see [1]). In case $p > 1$, $0 \leq \lambda < \frac{1}{p}$, F.Chiarenza and M.Frasca [7] showed the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and a singular integral operator in the Morrey spaces. Hence, in particular, it implies the boundedness of the Riesz transform in Morrey spaces. It means that, in case $p > 1$, $0 \leq \lambda < \frac{1}{p}$, for any $f \in M_{\lambda,p}$ we have $R_j f \in M_{\lambda,p}$, and there exist $C_{\lambda,p} > 0$ such that

$$\|R_j f\|_{M_{\lambda,p}} \leq C_{\lambda,p} \cdot \|f\|_{M_{\lambda,p}}$$

holds for all $f \in M_{\lambda,p}$.

In [12], the authors introduced a discrete analogue of Morrey spaces and studied their inclusion properties. For $m \in Z^d$ and $n \in N \cup \{0\}$ define $S_{m,n} = \{k \in Z^d : \|k - m\| \leq n\}$, where $\|k\| := \max\{|k_1|, ..., |k_d|\}$. Following standard conventions, we denote the cardinality of a set S by $|S|$. Then we have $|S_{m,n}| = (2n+1)^d$ for all $m \in Z^d$ and each

$n \in N \cup \{0\}$. Discrete Morrey spaces $m_{\lambda,p}$, $0 \le \lambda \le \frac{1}{p}$, $1 \le p < \infty$, consist of the sequences $h = \{h_n\}_{n \in Z^d}$ for which the following norm is finite

$$\|h\|_{m_{\lambda,p}} = \sup_{m \in Z^d} \sup_{n \in N \cup \{0\}} \left[|S_{m,n}|^{-\lambda} \left(\sum_{k \in S_{m,n}} |h_k|^p \right)^{1/p} \right].$$

In [2, 3, 13, 14, 15, 16] it was proved the boundedness of the discrete Hardy-Littlewood maximal operators, discrete Riesz potentials, discrete Hilbert transform and discrete Ahlfors-Beurling transforms on discrete Morrey spaces.

Note that if $h = \{h_n\}_{n \in Z^d} \in m_{\lambda,p}$, $1 \le p < \infty$, $0 \le \lambda < \frac{1}{p}$, then the series $\sum_{m \in Z^d,\, m \ne n} \frac{n_j - m_j}{|n-m|^{d+1}} \cdot h_m$ absolutely converges. Indeed, for each $n \in Z^d$ we have

$$\sum_{m \in Z^d,\, m \ne n} \frac{|n_j - m_j|}{|n-m|^{d+1}} \cdot |h_m| \le \sum_{j=1}^{\infty} \sum_{2^{j-1} \le \|n-m\| < 2^j} \frac{|h_m|}{|n-m|^d}$$

$$\le \sum_{j=1}^{\infty} \frac{1}{2^{d(j-1)}} \sum_{\|n-m\| < 2^j} |h_m| \le \sum_{j=1}^{\infty} \frac{1}{2^{d(j-1)}} \left(\sum_{\|n-m\| < 2^j} |h_m|^p \right)^{1/p} \cdot (2^{j+1} - 1)^{d - d/p}$$

$$\le 4^d \|h\|_{m_{\lambda,p}} \cdot \sum_{j=1}^{\infty} 2^{d(j+1)(\lambda - 1/p)} \le \frac{16\|h\|_{m_{\lambda,p}}}{1 - 2^{d(\lambda - 1/p)}}. \quad (1)$$

It follows that if $h = \{h_n\}_{n \in Z^d} \in m_{\lambda,p}$, $1 \le p < \infty$, $0 \le \lambda < \frac{1}{p}$, then the Riesz transform of the sequence h exists.

3. Boundedness of the discrete Riesz transform on discrete Morrey spaces

We next present the main results of this paper.

Theorem 1. *Let $1 < p < \infty$, $0 \le \lambda < 1/p$. For any $h \in m_{\lambda,p}$ we have $\widetilde{R}_j(h) \in m_{\lambda,p}$, and there exists $c_{\lambda,p} > 0$ such that*

$$\|\widetilde{R}_j(h)\|_{m_{\lambda,p}} \le c_{\lambda,p} \cdot \|h\|_{m_{\lambda,p}}$$

holds for all $h \in m_{\lambda,p}$.

Proof. We define the function $f(x)$ to be $\frac{2^d}{\gamma(d)} h_n$ for $x \in P(n, 1/4)$, $n \in Z^d$, and 0 elsewhere, where

$$P(n, \delta) := \{y \in R^d : \|y - n\| < \delta\}.$$

We first show that $f \in M_{\lambda,p}$. Indeed, for any $x \in P(n, 1/2)$, $n \in Z^d$ we have:

if $r \in (0, 1/4]$, then

$$|B(x;r)|^{-\lambda} \|f\|_{L_p(B(x;r))} = |\theta_{(d)} r^d|^{-\lambda} \left(\int_{B(x;r)} |f(y)|^p dy \right)^{1/p}$$

$$\leq |\theta_{(d)} r^d|^{1/p-\lambda} \cdot \frac{2^d}{\gamma_{(d)}} |h_n| \leq \frac{2^d \theta_{(d)}^{1/p-\lambda}}{\gamma_{(d)}} \|h\|_{m_{\lambda,p}}, \tag{2}$$

where $\theta_{(d)} = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$;

if $r \in (1/4, 1]$, then

$$|B(x;r)|^{-\lambda} \|f\|_{L_p(B(x;r))} \leq |\theta_{(d)} r^d|^{-\lambda} \left(\frac{1}{2^d} \sum_{k \in S_{n,1}} |\frac{2^d}{\gamma_{(d)}} h_k|^p \right)^{1/p}$$

$$\leq |\theta_{(d)} r^d|^{-\lambda} \cdot 2^{d-d/p} \cdot \frac{3^{d\lambda}}{\gamma_{(d)}} \|h\|_{m_{\lambda,p}} \leq \frac{2^{d-d/p} 3^{d\lambda}}{\theta_{(d)}^{\lambda} \gamma_{(d)}} \|h\|_{m_{\lambda,p}}; \tag{3}$$

if $r \in (m, m+1]$, $m \in N$, then

$$|B(x;r)|^{-\lambda} \|f\|_{L_p(B(x;r))} \leq |\theta_{(d)} r^d|^{-\lambda} \left(\frac{1}{2^d} \sum_{k \in S_{n,m+1}} \left| \frac{2^d}{\gamma_{(d)}} h_k \right|^p \right)^{1/p}$$

$$\leq |\theta_{(d)} r^d|^{-\lambda} \cdot 2^{d-d/p} \cdot \frac{(2m+3)^{d\lambda}}{\gamma_{(d)}} \|h\|_{m_{\lambda,p}} \leq \frac{2^{d-d/p} 5^{d\lambda}}{\theta_{(d)}^{\lambda} \gamma_{(d)}} \|h\|_{m_{\lambda,p}}. \tag{4}$$

From inequalities (2), (3), (4) it follows that $f \in M_{\lambda,p}$, and

$$\|f\|_{M_{\lambda,p}} = \sup_{x \in R^d} \sup_{r>0} \left[|B(x;r)|^{-\lambda} \|f\|_{L_p(B(x;r))} \right] \leq c_0 \cdot \|h\|_{m_{\lambda,p}},$$

where

$$c_0 = \max \left\{ \frac{2^{d-d/p} 5^{d\lambda}}{\theta_{(d)}^{\lambda} \gamma_{(d)}}, \frac{2^d \theta_{(d)}^{1/p-\lambda}}{\gamma_{(d)}} \right\}.$$

Therefore, $R_j f \in M_{\lambda,p}$ and

$$\|R_j f\|_{M_{\lambda,p}} \leq c_0 \cdot C_{\lambda,p} \|h\|_{m_{\lambda,p}}. \tag{5}$$

Define the function $F(x)$ to be $(\widetilde{R}_j h)_n$ for $x \in P(n, 1/2)$, $n \in Z^d$ and

$$G(x) = (R_j f)(x) - F(x). \tag{6}$$

We first show that $G \in M_{\lambda,p}$. For any $x \in P(n, 1/2)$, $n \in Z^d$ we have

$$G(x) = \sum_{m\in Z^d,\, m\neq n} 2^d h_m \int_{P(m,1/4)} \frac{x_j - y_j}{|x-y|^{d+1}} dy - \sum_{m\in Z^d,\, m\neq n} \frac{n_j - m_j}{|n-m|^{d+1}} \cdot h_m$$

$$+2^d h_n \cdot \text{v.p.} \int_{P(n,1/4)} \frac{x_j - y_j}{|x-y|^{d+1}} dy$$

$$= 2^d \sum_{m\in Z^d,\, m\neq n} h_m \int_{P(m,1/4)} \left(\frac{x_j - y_j}{|x-y|^{d+1}} - \frac{n_j - m_j}{|n-m|^{d+1}} \right) dy$$

$$+2^d h_n \cdot \text{v.p.} \int_{P(n,1/4)} \frac{x_j - y_j}{|x-y|^{d+1}} dy = G_1(x) + G_2(x). \tag{7}$$

Let $m \neq n$. Since for every $x \in P(n, 1/2)$ and $y \in P(m, 1/4)$

$$|n-m| - 3/4 \leq |x-y| \leq |n-m| + 3\sqrt{d}/4, \;\; |n_j - y_j| \leq |n-m| + 3/4,$$

then we get

$$\left| \frac{x_j - y_j}{|x-y|^{d+1}} - \frac{n_j - m_j}{|n-m|^{d+1}} \right| \leq \frac{|x_j - n_j|}{|x-y|^{d+1}}$$

$$+|n_j - y_j| \cdot \left| \frac{1}{|x-y|^{d+1}} - \frac{1}{|n-m|^{d+1}} \right| + \frac{|y_j - m_j|}{|n-m|^{d+1}} \leq \frac{c_1}{|n-m|^{d+1}},$$

where c_1 is a constant depending only d.

Therefore, for every $x \in P(n, 1/2)$

$$|G_1(x)| \leq 2^d \sum_{m\in Z^d,\, m\neq n} |h_m| \int_{P(m,1/4)} \left| \frac{x_j - y_j}{|x-y|^{d+1}} - \frac{n_j - m_j}{|n-m|^{d+1}} \right| dy$$

$$\leq \sum_{m\in Z^d,\, m\neq n} \frac{c_1 |h_m|}{|n-m|^{d+1}}. \tag{8}$$

From (8) and (1) it follows that

$$|G_1(x)| \leq \sum_{m\in Z^d,\, m\neq n} \frac{c_1 |h_m|}{|n-m|^{d+1}} \leq c_2 \|h\|_{m_{\lambda,p}}, \tag{9}$$

where $c_2 = \frac{16 c_1}{1 - 2^{d(\lambda - 1/p)}}$.

If $r \in (0, 1)$, then we have from (9)

$$|B(x;r)|^{-\lambda} \|G_1\|_{L_p(B(x;r))} = |\theta_{(d)} r^d|^{-\lambda} \left(\int_{B(x;r)} |G_1(y)|^p dy \right)^{1/p}$$

$$\leq c_2 |\theta_{(d)} r^d|^{1/p - \lambda} \|h\|_{m_{\lambda,p}} \leq c_3 \|h\|_{m_{\lambda,p}}, \tag{10}$$

where $c_3 = c_2 \cdot \theta_{(d)}^{1/p-\lambda}$;

if $r \in [k-1, k)$, $k \in N$, $k \geq 2$, then from (8) and from the Hölder's inequality we have

$$|B(x;r)|^{-\lambda} \|G_1\|_{L_p(B(x;r))} = |\theta_{(d)} r^d|^{-\lambda} \left(\int_{B(x;r)} |G_1(y)|^p dy \right)^{1/p}$$

$$\leq |\theta_{(d)} r^d|^{-\lambda} \left(\int_{P(n;k+1/2)} |G_1(y)|^p dy \right)^{1/p}$$

$$= |\theta_{(d)} r^d|^{-\lambda} \left(\sum_{s \in Z^d:\ \|s-n\| \leq k} \int_{P(s;1/2)} |G_1(y)|^p dy \right)^{1/p}$$

$$\leq |\theta_{(d)} r^d|^{-\lambda} \left(\sum_{s \in Z^d:\ \|s-n\| \leq k} \left(\sum_{m \in Z^d,\ m \neq s} \frac{c_1 |h_m|}{|s-m|^{d+1}} \right)^p \right)^{1/p}$$

$$\leq c_1 |\theta_{(d)} r^d|^{-\lambda} \left(\sum_{s \in Z^d:\ \|s-n\| \leq k} \left(\sum_{m \neq s} \frac{|h_m|^p}{|s-m|^{d+1}} \right) \cdot \left(\sum_{m \neq s} \frac{1}{|s-m|^{d+1}} \right)^{p-1} \right)^{1/p}$$

$$\leq c_1 c_4^{1-1/p} |\theta_{(d)} r^d|^{-\lambda} \left(\sum_{s \in Z^d:\ \|s-n\| \leq k} \sum_{m \neq s} \frac{|h_m|^p}{|s-m|^{d+1}} \right)^{1/p}$$

$$= c_1 c_4^{1-1/p} |\theta_{(d)} r^d|^{-\lambda} \left(\sum_{m \in Z^d} |h_m|^p \sum_{\|s-n\| \leq k, s \neq m} \frac{1}{|s-m|^{d+1}} \right)^{1/p}$$

$$\leq \frac{c_1 c_4^{1-1/p}}{|\theta_{(d)} r^d|^{\lambda}} \left(\sum_{\|m-n\| \leq 2k} |h_m|^p c_4 + \sum_{i=1}^{\infty} \sum_{2^i k < \|n-m\| \leq 2^{i+1} k} \frac{|h_m|^p}{k \cdot 2^{i(d+1)-3d-1}} \right)^{1/p}$$

$$\leq \frac{c_1 c_4^{1-1/p}}{|\theta_{(d)} r^d|^{\lambda}} \left(c_4 (2k+1)^{dp\lambda} \|h\|_{m_{\lambda,p}}^p + \sum_{i=1}^{\infty} \frac{(2^{i+1} k)^{dp\lambda} \|h\|_{m_{\lambda,p}}^p}{k \cdot 2^{i(d+1)-3d-1}} \right)^{1/p}$$

$$\leq \frac{c_1 c_4^{1-1/p}}{\theta_{(d)}^{\lambda}} \|h\|_{m_{\lambda,p}} \left(c_4 \cdot 5^{dp\lambda} + \frac{2^{2dp\lambda+3d+1}}{k \cdot (2^{d+1-dp\lambda} - 1)} \right)^{1/p} \leq c_5 \|h\|_{m_{\lambda,p}}, \qquad (11)$$

where

$$c_4 = \sum_{m \in Z^d:\, m \neq 0} \frac{1}{|m|^{d+1}} \leq \sum_{n \in N} \frac{(2n+1)^d - (2n-1)^d}{n^{d+1}} \leq d \cdot 3^{d-2} \pi^2,$$

$$c_5 = \frac{c_1 c_4^{1-1/p}}{\theta_{(d)}^{\lambda}} \left(c_4 \cdot 5^{dp\lambda} + \frac{2^{2dp\lambda+3d+1}}{k \cdot (2^{d+1-dp\lambda} - 1)} \right)^{1/p}.$$

It follows from (10), (11) that $G_1 \in M_{\lambda,p}$ and

$$\|G_1\|_{M_{\lambda,p}} = \sup_{x\in R^d} \sup_{r>0} \left[|B(x;r)|^{-\lambda} \|G_1\|_{L_p(B(x;r))}\right] \le c_6 \|h\|_{m_{\lambda,p}}, \tag{12}$$

where $c_6 = \max\{c_3, c_5\}$.

Let us show that $G_2 \in M_{\lambda,p}$.

For every $n \in Z^d$ and $a > 0$ denote

$$\Gamma(n;a) = \{y \in R^d : \|y-n\| = a\},$$

and for every $x \in P(n, 1/2)$ denote

$$\delta_x = \rho(x; \Gamma(n;1/4)) = \inf\{|x-y| : y \in \Gamma(n;1/4)\}.$$

For every $x \in P(n,1/2) \setminus \Gamma(n;1/4)$ we have

$$|G_2(x)| = 2^d |h_n| \left| \int_{P(n,1/4)} \frac{x_j - y_j}{|x-y|^{d+1}} dy \right|$$

$$\le 2^d |h_n| \int_{B(x,d)\setminus B(x;\delta_x)} \frac{dy}{|x-y|^d} = 2^d d\theta_{(d)} |h_n| \ln \frac{d}{\delta_x} \tag{13}$$

Since

$$\int_{P(n,1/2)} \left| \ln \frac{d}{\delta_x} \right|^p dx \le 4^d d \int_0^{1/4} \left| \ln \frac{d}{t} \right|^p dt,$$

then for every $n \in Z^d$

$$\int_{P(n,1/2)} |G_2(x)|^p \, dx \le (2^d d\theta_{(d)})^p |h_n|^p \int_{P(n,1/2)} \left| \ln \frac{d}{\delta_x} \right|^p dx \le c_7 |h_n|^p, \tag{14}$$

where

$$c_7 = 4^d d \cdot (2^d d\theta_{(d)})^p \int_0^{1/4} \left| \ln \frac{d}{t} \right|^p dt.$$

Let $x \in P(n,1/2)$. If $r \in (0,1)$, then it follows from (13) that

$$|B(x;r)|^{-\lambda} \|G_2\|_{L_p(B(x;r))} = |\theta_{(d)} r^d|^{-\lambda} \left(\int_{B(x;r)} |G_2(y)|^p dy \right)^{1/p}$$

$$\le |\theta_{(d)} r^d|^{-\lambda} \cdot 2^d d\theta_{(d)} \cdot \left[\sum_{\|s-n\|\le 1} |h_s| 4^d d \cdot r^{d-1} \int_0^{2r} \left| \ln \frac{d}{t} \right|^p dt \right]^{1/p}$$

$$\le (2d+1) |\theta_{(d)} r^d|^{-\lambda} \cdot 2^d d\theta_{(d)} \cdot \|h\|_{m_{\lambda,p}} (4^d d)^{1/p} \left[r^{d-1} \int_0^{2r} \left| \ln \frac{d}{t} \right|^p dt \right]^{1/p} \le c_8 \|h\|_{m_{\lambda,p}}, \tag{15}$$

where

$$c_8 = d(2d+1)\theta_{(d)}^{1-\lambda} \cdot 2^d (4^d d)^{1/p} \sup_{0<r<1} r^{-d\lambda} \left[r^{d-1} \int_0^{2r} \left| \ln \frac{d}{t} \right|^p dt \right]^{1/p} < \infty;$$

if $r \in [k-1, k)$, $k \in N$, $k \geq 2$, then it follows from (14) that

$$|B(x;r)|^{-\lambda} \|G_2\|_{L_p(B(x;r))} = |\theta_{(d)} r^d|^{-\lambda} \left(\int_{B(x;r)} |G_2(y)|^p dy \right)^{1/p}$$

$$\leq |\theta_{(d)} r^d|^{-\lambda} \left(\int_{P(n;k+1/2)} |G_2(y)|^p dy \right)^{1/p}$$

$$= |\theta_{(d)} r^d|^{-\lambda} \left(\sum_{s \in Z^d \colon \|s-n\| \leq k} \int_{P(s;1/2)} |G_2(y)|^p dy \right)^{1/p}$$

$$\leq |\theta_{(d)} r^d|^{-\lambda} \left(\sum_{s \in Z^d \colon \|s-n\| \leq k} c_7 |h_n|^p \right)^{1/p}$$

$$\leq \theta_{(d)}^{-\lambda} c_7^{1/p} (k-1)^{-d\lambda} (2k+1)^{d\lambda} \|h\|_{m_{\lambda,p}} \leq c_9 \|h\|_{m_{\lambda,p}}, \quad (16)$$

where $c_9 = 3^{d\lambda} c_7^{1/p} \theta_{(d)}^{-\lambda}$.

By (15), (16) we find that $G_2 \in M_{\lambda,p}$ and

$$\|G_2\|_{M_{\lambda,p}} = \sup_{x \in R^d} \sup_{r>0} \left[|B(x;r)|^{-\lambda} \|G_2\|_{L_p(B(x;r))} \right] \leq c_{10} \|h\|_{m_{\lambda,p}}, \quad (17)$$

where $c_{10} = \max\{c_8, c_9\}$.

Then it follows from (7), (12) and (17) that $G \in M_{\lambda,p}$ and

$$\|G\|_{M_{\lambda,p}} \leq (c_6 + c_{10}) \|h\|_{m_{\lambda,p}}. \quad (18)$$

Since $F(x) = G(x) + (R_j f)(x)$, by (5) and (18) we get that $F \in M_{\lambda,p}$ and

$$\|F\|_{M_{\lambda,p}} \leq (c_0 \cdot C_{\lambda,p} + c_6 + c_{10}) \|h\|_{m_{\lambda,p}}.$$

Therefore, for every $m \in Z^d$ and $n \in N \cup \{0\}$ we have

$$|S_{m,n}|^{-\lambda} \left(\sum_{k \in S_{m,n}} |(\widetilde{R}_j h)_k|^p \right)^{1/p} = (2n+1)^{-d\lambda} \left(\int_{P(m;n+1/2)} |F(x)|^p dx \right)^{1/p}$$

$$\leq (2n+1)^{-d\lambda} \|F\|_{L_p(B(m,\sqrt{d}(n+1/2)))} \leq (2n+1)^{-d\lambda} |B(m, \sqrt{d}(n+1/2))|^{\lambda} \|F\|_{M_{\lambda,p}}$$

$$\leq \theta_{(d)}^{\lambda}(c_0 \cdot C_{\lambda,p} + c_6 + c_{10})\|h\|_{m_{\lambda,p}}.$$

It follows that $\widetilde{R}_j(h) \in m_{\lambda,p}$ and

$$\|\widetilde{R}_j(h)\|_{m_{\lambda,p}} \leq \theta_{(d)}^{\lambda}(c_0 \cdot C_{\lambda,p} + c_6 + c_{10})\|h\|_{m_{\lambda,p}}.$$

This completes the proof of Theorem 1. ◄

References

[1] D.R. Adams, *Morrey Spaces*, Lecture Notes in Applied and Numerical Harmonic Analysis. Springer, Cham, 2015.

[2] R.A. Aliev, A.N. Ahmadova, *Boundedness of discrete Hilbert transform on discrete Morrey spaces*, Ufa Math. J., **13(1)**, 2021, 98–109.

[3] R.A. Aliev, A.N. Ahmadova, A.F. Huseynli, *Boundedness of the discrete Ahlfors-Beurling transform on discrete Morrey spaces*, Proc. Inst. Math. and Mech., **48(1)**, 2022, 123–131.

[4] R.A. Aliev, Kh.I. Nabiyeva, *The A-integral and restricted Riesz transform*, Constructive Mathematical Analysis, **3(3)**, 2020, 104-112.

[5] P. Auscher, T. Coulhon, X.T. Duong, S. Hoffman, *Riesz transform on manifolds and heat kernel regularity*, Ann. Sci. Ecole Norm. Sup., **37**, 2004, 911–957.

[6] A.P. Calderon, A. Zygmund, *On the existence of certain singular integrals*, Acta Mathematica, **88**, 1952, 85-139.

[7] F. Chiarenza, M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. Appl., **7(3-4)**, 1987, 273–279.

[8] T. Coulhon, X.T. Duong, *Riesz transform for* $1 \leq p \leq 2$, Trans. of Amer. Math. Soc., **351(3)**, 1999, 1151–1169.

[9] L. De Michele, G. Mauceri, *L_p multipliers on the Heisenberg group*, Michigan Math. J., **26**, 1979, 361–371.

[10] M. Dosso, I. Fofana, M. Sanogo, *On some subspaces of Morrey-Sobolev spaces and boundedness of Riesz integrals*, Ann. Polon. Math., **108(2)**, 2013, 133-153.

[11] H. Gunawan, D.I. Hakim, K.M. Limanta, A.A. Masta, *Inclusion properties of generalized Morrey spaces*, Math. Nachr., **290(2-3)**, 2017, 332-340.

[12] H. Gunawan, C. Schwanke, *The Hardy-Littlewood maximal operator on discrete Morrey spaces*, Mediterr. J. Math., **16(1)**, 2019, Article: 24.

[13] X.B. Hao, B.D. Li, S. Yang, *The Hardy–Littlewood maximal operator on discrete weighted Morrey spaces*, Acta Math. Hungar., **172**, 2024, 445–469.

[14] X.B. Hao, B.D. Li, S. Yang, *Estimates of discrete Riesz potentials on discrete weighted Lebesgue spaces*, Ann. Funct. Anal., **15**, 2024, Article: 51.

[15] X.B. Hao, B.D. Li, S. Yang, *Discrete Riesz Potentials on Discrete Weighted Morrey Spaces*, 2023, arXiv:2310.08458.

[16] M. Jakfar, M.A. Lukito, Sh. Fiangga, *Inner Products on Discrete Morrey Spaces*, European Journal of Pure and Applied Mathematics, **16(1)**, 2023, 144-155

[17] H. Jizheng, *The boundedness of Riesz transforms for Hermite expansions on the Hardy spaces*, J. of Math. Anal. and Appl., **385(1)**, 2012, 559-571.

[18] B. Langowski, A. Nowak, *On Derivatives, Riesz Transforms and Sobolev Spaces for Fourier–Bessel expansions*, J. of Fourier Anal. and Appl., **28**, 2022, Article: 1.

[19] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43(1)**, 1938, 126-166.

[20] E. Nakai, T. Yoneda, *Riesz transforms on generalized Hardy spaces and a uniqueness theorem for the Navier-Stokes equations*, Hokkaido Math. J., **40(1)**, 2011, 67-88.

[21] M. Ruzhansky, D. Suragan, N. Yessirkegenov, *Hardy-Littlewood, Bessel-Riesz, and fractional integral operators in anisotropic Morrey and Campanato spaces*, Fractional Calculus and Applied Analysis, **21**, 201), 577–612.

[22] E. Russ, $H_1 - L_1$ *boundedness of Riesz transforms on Riemannian manifolds and on graphs*, Potential Anal., **14**, 2001, 301–330.

[23] W. Sickel, *Smoothness spaces related to Morrey spaces - a survey*, I, Eurasian Math. J., **3(3)**, 2012, 110-149.

[24] W. Sickel, *Smoothness spaces related to Morrey spaces - a survey*, II, Eurasian Math. J., **4(1)**, 2013, 82-124.

[25] W. Sickel, D.C. Yang, W. Yuan, *Morrey and Campanato Meet Besov, Lizorkin and Triebel*, Springer, Berlin, 2010.

[26] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton, University Press, 1970.

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