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On the Basicity of double System of Exponents in the Weighted Lebesgue Space

S.R. Sadigova, A.I. Mirzabalayeva, M.H. Orujova*

Abstract. This paper considers double system of exponentials with linear phase in the weighted space $L_{p,\rho}$ with power weight $\rho(\cdot)$ on the segment $[-\pi,\pi]$. Under certain conditions on the weight function $\rho(\cdot)$ and on the perturbation parameters, the completeness, minimality and basicity of this system in $L_{p,\rho}$ is proved. An explicit expression for the biorthogonal system in the case of minimality is derived and its integral representation is obtained. The obtained results generalize all previously known results in this direction.

Key Words and Phrases: exponential system, basicity, weighted space.

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1. Introduction

Investigation of many partial differential equation by the application of Fourier method reduces to perturbed trigonometric system of sines (or cosines) of the form

$$\{\sin(nt+\alpha(t))\}_{n\in N}, \tag{1}$$

where $\alpha:\,[0,\pi]\to R$ is some function (N is a set of natural numbers). Similar problems were studied, for example, in the papers [2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 42]. To justify the Fourier method it is necessary to study the basicity properties (completeness, minimality, basis property, etc.) of these systems in different functional spaces. Complex versions of these systems are perturbed system of exponents of the form

$$\left\{e^{i(nt+\beta(t)sign\,n)}\right\}_{n\in Z}, \tag{2}$$

where $\beta:\,[-\pi,\pi]\to R$ is some function (Z is the set of integers). Basis properties of the systems (1) and (2) in corresponding spaces are closely linked, in Lebesgue spaces L_p they are well studied by various mathematicians (see, for example [2, 3, 5, 6, 7, 12, 13, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27]). The case $L_\infty = C[-\pi,\pi]$ is treated in [32]. In connection with application to solution of differential equations, the interest in Lebesgues spaces

*Corresponding author.

$L_{p(\cdot)}$ with variable summability power $p(\cdot)$ and in Morrey spaces $L^{p,\alpha}$ greatly increased in recent years. Problems of approximation in these spaces have also begun to be studied and basicity problems of the systems (1), (2) in $L_{p(\cdot)}$ are studied in [28, 43], and basicity of the classical system of exponents with linear phase in Morrey spaces are studied in [38, 44, 45]. Note that study of basicity properties of the systems (1), (2) in weighted spaces $L_{p,\rho}$ is equivalent to the study of analogous properties of the systems (1), (2) with corresponding degenerate coefficients in the spaces L_p. For this reason, it can be assumed that the study of the basicity of trigonometric systems in weighted Lebesgue spaces takes its origin from the paper of K.Babenko [18]. Later this area was developed in the works [14, 15, 16, 29, 30, 39, 40, 41]. The problem of basicity of the exponential system in the weighted space $L_{p,\rho} \equiv L_{p,\rho}(-\pi,\pi)$, $1 < p < +\infty$, is solved in the paper [31]. Such a condition is a Muckenhaupt condition with respect to the weight function $\rho(\cdot)$:

$$\sup_I \left(\frac{1}{|I|}\int_I \rho(t)\,dt\right)\left(\frac{1}{|I|}\int_I \rho^{-\frac{1}{p-1}}dt\right)^{p-1} < \infty, \tag{3}$$

where supis taken over all intervals $I \subset [-\pi,\pi]$ and $|I|$ is the length of the interval I.

In the papers [3, 15] the system (2) is considered in the case when $\beta(t) = \beta t$, where $\beta \in R$ is some real parameter and its basicity in $L_{p,\rho}$, $1 < p < +\infty$, is studied when $\rho(\cdot)$ has the following form

$$\rho(t) = \prod_{k=-r}^{r} |t - t_k|^{\alpha_k},$$

where $-\pi = t_{-r} < t_{-r+1} < ... < t_r < \pi$.

The class of weights, satisfying the condition (3), is denoted by A_p. It is easy to see that

$$\rho \in A_p \Leftrightarrow -1 < \alpha_k < p-1, \;\; k = \overline{-r,r}.$$

In this paper the minimality of the exponential system

$$\left\{e^{i\left(n+\frac{\beta}{2} sign\, n\right)t}\right\}_{n\in Z},$$

in the weighted space $L_{p,\rho}$, $1 < p < +\infty$, where $\beta \in C$ is a complex parameter, is studied. In contrast to the paper [3], an explicit expression of the biorthogonal system is built and its integral representation is obtained.

2. Preliminaries. Main lemma

Consider the following double system of exponents

$$\left\{e^{i[(n+\beta_1)t+\gamma]}; e^{-i[(k+\beta_2)t+\gamma_2]}\right\}_{n\in Z_+;k\in N}, \tag{4}$$

where $\beta_k = Re\beta_k + iIm\beta_k$, $\gamma_k = Re\,\gamma_k + iIm\,\gamma_k$, $k = 1,2$, are complex parameters, $Z_+ = \{0\} \bigcup N$. We assume that the weight function $\rho(\cdot)$ is of the following power form

$$\rho(t) = \prod_{k=-r}^{r} |t - t_k|^{\alpha_k},$$

where $-\pi = t_{-r} < t_{-r+1} < ... < t_0 = 0 < ... < t_r < \pi$, $\{\alpha_k\}_{k=\overline{-r,r}} \subset R$ are some numbers. We consider the weighted space $L_{p,\rho}$, $1 < p < +\infty$, with the norm $\|\cdot\|_{p,\rho}$:

$$\|f\|_{p,\rho} = \left(\int_{-\pi}^{\pi} |f(t)|^p \rho(t)\, dt \right)^{1/p}.$$

It is easy to see that basicity properties of the system (4) in $L_{p,\rho}$ are equivalent to basicity properties of the system

$$\left\{ e^{i(n+\beta_1)t}; e^{-i(k+\beta_2)t} \right\}_{n \in Z_+; k \in N}, \tag{5}$$

in $L_{p,\rho}$. We put $g(t) = e^{\frac{i}{2}(\beta_2 - \beta_1)t}$. It is evident that $\exists \delta > 0$:

$$0 < \delta \le |g(t)| \le \delta^{-1} < +\infty, \ \forall t \in [-\pi, \pi].$$

Multiplying the system (5) to the function $g(t)$, we immediately obtain from here that the basicity properties of the system (5) on $L_{p,\rho}$ are equivalent to the basicity properties of the following system

$$\left\{ e^{i\left(n + \frac{\beta}{2} sign\, n\right)t} \right\}_{n \in Z}, \tag{6}$$

on $L_{p,\rho}$, $\beta = \beta_1 + \beta_2$. Thus, the study of basicity properties of the system (4) on $L_{p,\rho}$ is reduced to the investigation of corresponding properties with respect to the system (6) on $L_{p,\rho}$.

Let $\beta \in C$ be some complex number. We will assume throughout the paper that $(1+z)^\beta$ is some fixed branch of multivalued analytic function $(1+z)^\beta$ on the complex plane with the cut along the semiline $(-\infty, -1) \subset R$ on the real axis and take

$$(1+z)^{-\beta} = \frac{1}{(1+z)^\beta}.$$

Analogously, we define a branch z^β of a multivalued function z^β on C with the cut along $(-\infty, 0) \subset R$ and $z^{-\beta} = \frac{1}{z^\beta}$.

We will essentially use the following main lemma in the proof of main results.

Lemma 1. *Let $Re\beta > -1$. Then the following Cauchy integral formulas hold*

$$J_m^-(z) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i(\beta+m)\theta} \left(1 + e^{i\theta}\right)^\beta}{e^{i\theta} - z} d\theta \equiv \begin{cases} 0, & |z| < 1, \\ -z^{-m-\beta-1} (1+z)^\beta, & |z| > 1, \end{cases}$$

$$J_m^+(z) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i(m+1)\theta} \left(1+e^{i\theta}\right)^{\beta}}{e^{i\theta}-z} d\theta \equiv \begin{cases} 0, & |z|>1\,, \\ z^m (1+z)^{\beta}\,, & |z|<1\,, \end{cases}$$

$$\forall m \in Z_+.$$

Proof. Consider an expression $J_m^+(z)$. Make the change of variables

$$tg\frac{\theta}{2} = t \Rightarrow e^{i\theta} = \frac{1+it}{1-it}.$$

We have

$$J_m^+(z) = \int_{-\infty}^{+\infty} \frac{(1+it)^{m+1}(1-it)^{-m-1}}{\frac{2^{-\beta}}{(1-it)^{-\beta}}\left(\frac{1+it}{1-it}-z\right)} \frac{2dt}{(1-it)(1+it)} =$$

$$= 2^{\beta+1} \int_{-\infty}^{+\infty} (1-it)^{-\beta} (1-it)^{-m-1} (1+it)^m \left[1+it-z+izt\right]^{-1} dt =$$

$$= \frac{2^{\beta+1}}{i(1+z)} \int_{-\infty}^{+\infty} (1-it)^{-\beta} (1-it)^{-m-1} \left[t-\frac{i(1-z)}{1+z}\right]^{-1} (1+it)^m\, dt.$$

Let $Rez = x,\ \ Imz = y$. We obtain

$$Im\left(i\frac{1-z}{1+z}\right) = \frac{1-x^2-y^2}{(1+x)^2+y^2} > 0, \qquad |z|<1,$$

and it is evident that

$$Im\left(i\frac{1-z}{1+z}\right) < 0, \qquad |z|>1.$$

Denote the integrand function by $F(w),\ \ w \in C$:

$$F(w) = (1-iw)^{-\beta-m-1} (1+iw)^m \left(w - i\frac{1-z}{1+z}\right)^{-1}.$$

It is obvious that for large values of $|w|$ the following estimation holds

$$|F(w)| \le \frac{M}{|w|^{2+Re\beta}},$$

where $M>0$ is some constant. Applying Theorem 5.3 from monograph [1] (see p. 127), we obtain that

$$J_m^+(z) = \frac{2^{\beta+1}}{i(1+z)} 2\pi i \operatorname*{Res}_{t=i\frac{1-z}{1+z}} \left[(1-it)^{-\beta-m-1}(1+it)^m \left(t-i\frac{1-z}{1+z}\right)^{-1}\right] =$$

$$= \frac{2^{\beta+2}\pi}{1+z} \left(\frac{2}{1+z}\right)^{-\beta-m-1} \left(\frac{2z}{1+z}\right)^m = 2\pi z^m (1+z)^{\beta}, \ \ |z|<1,$$

since for $|z| < 1$, the only pole of the function $F(w)$ in the upper half-plane is $w = i\frac{1-z}{1+z}$. Analogous reasoning implies that $J_m^+(z) \equiv 0$, $|z| > 1$, since for $|z| > 1$ the function $F(w)$ has no poles in the upper half-plane.

The formula for $J_m^-(z)$ is proved in a similar way.

The lemma is proved.

3. Minimality in $L_{p,\rho}$

Consider the following system of functions

$$\vartheta_n^+(t) = \frac{e^{-i\frac{\beta}{2}t}}{2\pi}\left(1+e^{it}\right)^\beta \sum_{k=0}^n C_{-\beta}^{n-k} e^{-ikt}, \quad n \in Z_+;$$
$$\vartheta_m^-(t) = -\frac{e^{-i\frac{\beta}{2}t}}{2\pi}\left(1+e^{it}\right)^\beta \sum_{k=1}^m C_{-\beta}^{m-k} e^{ikt}, \quad m \in N;$$

where

$$C_{-\gamma}^k = \frac{\gamma(\gamma-1)\ldots(\gamma-k+1)}{k!},$$

is a binomial coefficient. Accordingly, we denote

$$e_n^+(t) \equiv e^{i\left(n+\frac{\beta}{2}\right)t}, \quad n \in Z_+; \quad e_k^-(t) \equiv e^{-i\left(n+\frac{\beta}{2}\right)t}, \quad k \in N.$$

Assume that $Re\beta > -1$. The expansion in powers of z of the function $(1+z)^{-\beta} J_m^+(z)$ that is analytic on $|z| < 1$ is

$$(1+z)^{-\beta} J_m^+(z) = \sum_{n=0}^{\infty} a_{n;m}^+ z^n,$$

where

$$a_{n;m}^+ = \int_{-\pi}^{\pi} e^{i\left(m+\frac{\beta}{2}\right)t} \vartheta_n^+(t)\,dt.$$

On the other hand, it follows from Lemma 1 that

$$(1+z)^{-\beta} J_m^+(z) \equiv z^m, \quad |z| < 1.$$

Comparing the corresponding coefficients, we arrive at the following equalities

$$\int_{-\pi}^{\pi} e_m^+(t)\,\vartheta_n^+(t)\,dt = \delta_{nm}, \quad \forall n, m \in Z_+.$$

Expanding the function $(1+z)^{-\beta} J_m^+(z)$ at infinity in powers of z^{-1}, we obtain

$$(1+z)^{-\beta} J_m^+(z) = \sum_{n=1}^{\infty} b_{n;m}^+ z^{-n}, \quad |z| > 1,$$

where
$$b_{n;m}^{+} = \int_{-\pi}^{\pi} e^{i\left(m+\frac{\beta}{2}\right)t} \vartheta_n^{-}(t)\, dt, \quad m \in Z_+, \quad n \in N.$$
It is easy to see that
$$\lim_{|z|\to\infty} (1+z)^{-\beta} J_m^{+}(z) = 0.$$
On the other hand, again, as follows from Lemma 1, we have
$$(1+z)^{-\beta} J_m^{+}(z) \equiv 0, \quad |z| > 1.$$
These two expansions imply
$$\int_{-\pi}^{\pi} e^{i\left(m+\frac{\beta}{2}\right)t} \vartheta_n^{-}(t)\, dt = 0, \quad \forall m \in Z_+, \quad \forall n \in N.$$
The relations
$$\begin{array}{l} \int_{-\pi}^{\pi} e_m^{-}(t)\, \vartheta_n^{+}(t)\, dt = 0, \quad m \in N, \quad n \in Z_+; \\ \int_{-\pi}^{\pi} e_m^{-}(t)\, \vartheta_n^{-}(t)\, dt = \delta_{nm}, \qquad \forall n, m \in N \end{array}$$
can be proved analogously. As a result, we obtain the validity of the following statement.

Proposition 1. *Let $Re\beta > -1$. Then for all admissible values of indices n and m the following relations*
$$\int_{-\pi}^{\pi} e_n^{\pm}(t)\, \vartheta_m^{\pm}(t)\, dt = \delta_{nm}, \qquad \int_{-\pi}^{\pi} e_n^{\pm}(t)\, \vartheta_m^{\mp}(t)\, dt = 0,$$
hold.

Consider the following proposition.

Proposition 2. *Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(L_{p,\rho})^* = L_{q,\rho}$ and every functional $\vartheta^* \in (L_{p,\rho})^*$ is represented, by the uniquely determined for it function $\vartheta \in L_{q,\rho}$, by the following expression*
$$\vartheta^*(f) = \int_{-\pi}^{\pi} f \bar{\vartheta} \rho dt, \quad \forall f \in L_{p,\rho}.$$

Following this proposition, define the following system of functions
$$h_n^{\pm}(t) = \rho^{-1}(t)\, \overline{\vartheta_n^{\pm}(t)}.$$
It is easy to see that the system $\{h_n^{\pm}\}$ belongs to the space $L_{q,\rho}$ when
$$\alpha_k < \frac{1}{q-1}, \;\; k = \overline{-r+1, r-1}; \;\; Re\beta - \frac{\alpha_{\pm r}}{p} > -\frac{1}{q}.$$
This follows directly from the representation of $\{\vartheta_n^{\pm}\}$ and from the relation
$$\int_{-\pi}^{\pi} \left|h_n^{\pm}\right|^q \rho dt = \int_{-\pi}^{\pi} \rho^{1-q} \left|\vartheta_n^{\pm}\right|^q dt.$$
Taking into account that $\frac{1}{q-1} = \frac{p}{q}$, we obtain the following theorem from Proposition 1 and 2.

Theorem 1. *Assume that the following inequalities hold*

$$Re\beta > -1; \quad -1 < \alpha_k < \tfrac{p}{q}, \quad k = \overline{-r+1, r-1};$$

$$-1 < \alpha_{\pm r} < \tfrac{p}{q} + pRe\beta.$$

Then the exponential system $\left\{e^{i\left(n+\frac{\beta}{2}sign\, n\right)t}\right\}_{n\in Z}$ *is minimal in* $L_{p,\rho}$, $1 < p < +\infty$.

The following lemma plays a very important role in the study of orthogonal series.

Lemma 2. *The system* $\{\vartheta_n^{\pm}\}$ *has the following integral representation*

$$\vartheta_{n-1}^{+}(t) = \frac{e^{-i\left(n+\frac{\beta}{2}\right)t}}{2\pi}\left(1+e^{it}\right)^{\beta}\left[\left(1+e^{it}\right)^{-\beta} - \frac{\sin\pi\beta}{\pi}e^{i(t+\pi)n}\int_0^1 \frac{x^{n+\beta-1}}{1+xe^{it}}dx\right],$$

$$\vartheta_{n}^{-}(t) = \frac{e^{i\left(n-\frac{\beta}{2}\right)t}}{2\pi}\left(1+e^{it}\right)^{\beta}\left[\left(1+e^{-it}\right)^{-\beta} - \frac{\sin\pi\beta}{\pi}e^{i(\pi-t)n}\int_0^1 \frac{x^{n+\beta-1}(1-x)^{-\beta}}{1+xe^{it}}dx\right], \quad n\in N,$$

for $Re\beta \in (-1,1)$.

Proof. We will prove this lemma with regard to ϑ_n^- since for ϑ_n^+ it is proved in exactly the same way. Thus, let

$$\vartheta_n^-(t) = -\frac{e^{-i\frac{\beta}{2}t}}{2\pi}\left(1+e^{it}\right)^{\beta}\sum_{k=1}^{n} C_{-\beta}^{n-k}e^{itk}.$$

Make the following transformation

$$\sum_{k=1}^{n} C_{-\beta}^{n-k}e^{ikt} = e^{int}\sum_{k=0}^{n-1} C_{-\beta}^{k}e^{-ikt} = e^{int}\left[\left(1+e^{-it}\right)^{-\beta} - \sum_{k=n}^{\infty} C_{-\beta}^{k}e^{-ikt}\right] =$$

$$= e^{int}\left[\left(1+e^{-it}\right)^{-\beta} - e^{-int}\sum_{k=0}^{\infty} C_{-\beta}^{k+n}e^{-ikt}\right] =$$

$$= e^{int}\left[\left(1+e^{-it}\right)^{-\beta} - e^{-int}\frac{(-1)^n(\beta)_n}{n!}F\left(1; n+\beta; n+1; e^{i(\pi-t)}\right)\right],$$

where

$$(\beta)_n = \beta(\beta+1)\ldots\ldots(\beta+n-1) = \frac{\Gamma(\beta+n)}{\Gamma(\beta)},$$

$\Gamma(\cdot)$ is Euler's gamma function and $F(a;b;c;z)$ is hypergeometric function.

Using integral representation for hypergeometric functions (see [46], p. 72), we find from here that

$$\sum_{k=1}^{n} C_{-\beta}^{n-k} e^{ikt} = e^{int} \left[\left(1+e^{-it}\right)^{-\beta} - e^{i(\pi-t)n} \frac{\sin \pi\beta}{\pi} \int_0^1 \frac{x^{n+\beta-1}(1-x)^{-\beta}}{1+xe^{-it}} dx \right].$$

Substituting this representation into the expression of ϑ_n^-, we arrive at the required fact.
The lemma is proved.

4. Completeness in $L_{p,\rho}$

The following lemma on the uniform convergence plays an important role in the study of the completeness of the exponential system (6) in $L_{p,\rho}$.

Lemma 3. *Let $-1 < Re\beta < 0$ and $\beta = 0$. If $\psi(\cdot)$ is an arbitrary Holder function on $[-\pi, \pi] : e^{i\beta\pi}\psi(-\pi) = \psi(\pi) = 0$, then the series*

$$\sum_{n=0}^{\infty} a_n^+ e_n^+(t) + \sum_{n=1}^{\infty} a_n^- e_n^-(t),$$

uniformly converges to $\psi(\cdot)$ on $[-\pi, \pi]$, where $a_n^\pm = \int_{-\pi}^{\pi} \psi(t) h_n^\pm(t) \rho(t) dt$.

Proof. Consider the following conjugate problem: find a piecewise analytic function $F(z)$ inside and outside of the unit circle, which the boundary values on the unit circle satisfy the following condition

$$F^+\left(e^{it}\right) + e^{-i\beta t} F^-\left(e^{it}\right) = e^{-i\frac{\beta}{2}t}\psi(t), \ \ t \in (-\pi, \pi]. \tag{7}$$

We will solve this problem by the method developed in the monograph F.D. Gakhov [1, page 427]. Consider the following multi-valued analytic function in the complex plane

$$\omega(z) = (z+1)^\gamma.$$

We carry out cut on the plane z from zero to infinity $(-\infty)$ along the negative real axis. In the cut plane like that, this function will be unique, and the incision for it will be a line of discontinuity. Denote this branch by

$$\omega_{-1}(z) = (z+1)_{-1}^\gamma.$$

Let us define

$$\gamma = \frac{1}{2\pi i} \ln e^{-i2\beta\pi} \Rightarrow Re\gamma = -Re\beta.$$

A solution of problem (7) is the following Cauchy type integral

$$F^+(z) = (z+1)_{-1}^\gamma X_1^+(z)\Psi^+(z); F^-(z) = \left(\frac{z+1}{z}\right)_{-1}^\gamma X_1^-(z)\Psi^-(z),$$

where

$$X_1(z) = \exp[\Gamma(z)],$$

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\ln[\tau^{-\gamma} G(\arg\tau)]}{\tau - z} d\tau,$$

$$\Psi(z) = \frac{1}{2\pi i}\int_L \frac{(\tau+1)_{-1}^{-\gamma} \varphi(\arg\tau)}{X_1^+(\tau)(\tau - z)} d\tau,$$

$$G(t) = e^{-i\beta t};\ \varphi(t) = e^{-i\frac{\beta}{2}t}\psi(t),$$

$L-$ is a unit circle, which goes around from the point $e^{-i\pi}$ to the point $e^{i\pi}$ in the positive direction.

The fact that $F(z)$ satisfies the boundary condition (7), follows directly from the Sokhotskii-Plemelj formulas. Let $0 < Re\gamma < 1$. It is clear that the function $G(t)$ satisfies the Holder condition on the interval $[-\pi, \pi]$. Moreover, it is easy to verify that the function $\tau^{-\gamma} G(\arg\tau)$ is continuous at a point $\tau = -1$, and as a result it satisfies a certain Holder condition on the unit circle. Then according to the results of the monograph F.D.Gakhov [1, page 55] the function $X_1^{\pm}(\tau)$ satisfies the Holder condition on L. Denote

$$L_{-\pi} = \left\{ z = e^{it} : t \in \left[-\pi, -\frac{\pi}{2}\right] \right\}.$$

Assume

$$\varphi^*(\tau) = \frac{\psi(\arg\tau)}{X^+(\tau)\,\tau^{\frac{\beta}{2}}}, \qquad \tau \in L.$$

Let $\left[(z+1)^{-\gamma}\right]^*$ be a branch, that is holomorphic on the cut along $L_{-\pi}$ on the plane of the function $(z+1)^{-\gamma}$, having the values $(t+1)_{-1}^{-\gamma}$ on the left side of $L_{-\pi}$. So $\left[(t+1)^{-\gamma}\right]^* = (t+1)_{-1}^{-\gamma}$ on $L_{\pi} = \left\{ z = e^{it} : t \in \left[\frac{\pi}{2}, \pi\right] \right\}$,
using the results of the monograph [1] (see page 74), the function $\Psi(z)$ in the vicinity of the point $z = -1$ on the contour L can be represented as

$$\Psi(t) = \left[\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0) - \frac{ctg\,\gamma\pi}{2i}\varphi^*(-1-0)\right]\frac{1}{[(t+1)^{\gamma}]^*} + \Phi(t), \text{ for } t \in L_{\pi}; \quad (8)$$

$$\Psi(t) = \left[\frac{ctg\,\gamma\pi}{2i}\varphi^*(-1+0) - \frac{e^{-i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1-0)\right]\frac{1}{[(t+1)^{\gamma}]^*} + \Phi(t) \text{ for } t \in L_{-\pi}, \quad (9)$$

where under $\left[(t+1)^{-\gamma}\right]^*$ we mean the limit of the function $\left[(z+1)^{-\gamma}\right]^*$, when z tends to t on the left of $L_{-\pi} \bigcup L_{\pi}$, and moreover

$$\Phi(t) = \frac{\Phi^*(t)}{|t+1|^{\gamma_0}}, \ \gamma_0 < Re\,\gamma, \tag{10}$$

and the function $\Phi^*(t)$ belongs to the Holder class at the neighborhood of the point $z=-1$.

Applying the Sokhotskii-Plemelj formula from these representations near the point $z=-1$ we have

$$F^+(t)=(t+1)_{-1}^{\gamma}X_1^+(t)\left[\frac{1}{2}(t+1)_{-1}^{-\gamma}\varphi^*(t)+\frac{1}{2\pi i}\int_L\frac{\varphi^*(\tau)}{(\tau+1)_{-1}^{\gamma}(\tau-t)}d\tau\right]=$$

$$=X_1^+(t)\left[\frac{1}{2}\varphi^*(t)+\right.$$

$$\left.+\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0)-\frac{ctg\,\gamma\pi}{2i}\varphi^*(-1-0)\right]+(t+1)_{-1}^{\gamma}X_1^+(t)\,\Phi(t).$$

Passing to the limit as $t\to-1-0$, and taking into account the relation (10), we obtain

$$F^+(-1-0)=X_1^+(-1)\left[\frac{1}{2}\varphi^*(-1-0)+\frac{e^{i\gamma\pi}}{2i\sin\,\gamma\pi}\ \varphi^*(-1+0)-\frac{ctg\,\gamma\pi}{2i}\varphi^*(-1-0)\right]=$$

$$=X_1^+(-1)\left[\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0)-\frac{e^{-i\gamma\pi}}{2i\sin\gamma\pi}\ \varphi^*(-1-0)\right].$$

Similarly, from the expressions (8) and (9) we obtain

$$F^+(-1+0)=X_1^+(-1)\left[\frac{1}{2}\varphi^*(-1+0)+\frac{ctg\gamma\pi}{2i}\varphi^*(-1+0)-\frac{e^{-i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1-0)\right]=$$

$$=X_1^+(-1)\left[\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0)-\frac{e^{-i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1-0)\right].$$

Thus, $F^+(-1-0)=F^+(-1+0)$, i.e. $F^+(t)$ is continuous at the point $z=-1$, and as a result, it satisfies a certain Holder condition on L. Expanding $F^+(z)$ on z at zero, we obtain

$$F^+(z)=\sum_{n=0}^{\infty}a_n^+z^n,$$

where

$$a_n^+=\int_{-\pi}^{\pi}\psi(t)\,h_n^+(t)\,\rho(t)\,dt,\quad n\in Z_+.$$

We have

$$\frac{1}{2\pi i}\int_{|z|=r<1}F^+(z)\,z^{-n-1}dz=\begin{cases}a_n^+, & n\geq 0,\\ 0, & n<0.\end{cases}$$

Passing to the limit as $r\to 1-0$, hence we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}F^+\left(e^{it}\right)e^{-int}dt=\begin{cases}a_n^+, & n\geq 0\,,\\ 0, & n<0.\end{cases}$$

As, the function $F^{+}\left(e^{it}\right)$ satisfies Holder condition on L, then its Fourier series on classical system of exponents $\left\{e^{int}\right\}_{n\in Z}$ uniformly converges to it on $[-\pi,\pi]$, and consequently

$$F^{+}\left(e^{it}\right)=\sum_{n=0}^{\infty}a_n^{+}e^{int},\quad t\in[-\pi,\pi].$$

Now, we investigate the boundary properties of the function $F^{-}(z)$. Similarly to the case $F^{+}(z)$, using the representation (8) - (10), and the Sohotskogo- Plemelj formula, we obtain

$$F^{-}(t)=t_{-1}^{-\gamma}(1+t)_{-1}^{\gamma}X_1^{-}(t)\Psi^{-}(t)=t_{-1}^{-\gamma}(1+t)_{-1}^{\gamma}X_1^{-}(t)$$
$$\left\{-\frac{1}{2}\varphi^{*}(t)(1+t)_{-1}^{-\gamma}+\left[\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^{*}(-1+0)-\frac{ctg\gamma\pi}{2i}\varphi^{*}(-1-0)\right](1+t)_{-1}^{-\gamma}+\Phi(t)\right\}=$$

$$=t_{-1}^{-\gamma}X_1^{-}(t)\left[-\frac{\varphi^{*}(t)}{2}+\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^{*}(-1+0)-\frac{ctg\gamma\pi}{2i}\varphi^{*}(-1-0)+(1+t)_{-1}^{\gamma}\Phi(t)\right].$$

Passing to the limit as $t\to 1-0$, we have

$$F^{-}(-1-0)=e^{-i\gamma\pi}X_1^{-}(-1)\left[-\frac{\varphi^{*}(-1-0)}{2}+\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^{*}(-1+0)-\frac{e^{i\gamma\pi}}{2i\ \sin\gamma\pi}\varphi^{*}(-1-0)\right]=$$

$$=e^{-i\gamma\pi}X_1^{-}(-1)\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\left[\varphi^{*}(-1+0)-\varphi^{*}(-1-0)\right]=0.$$

$$F^{-}(-1+0)=e^{i\gamma\pi}X_1^{-}(-1)\left[-\frac{\varphi^{*}(-1+0)}{2}+\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^{*}(-1+0)-\frac{ctg\gamma\pi}{2i}\varphi^{*}(-1-0)\right]=$$

$$=e^{i\gamma\pi}X_1^{-}(-1)\frac{ctg\gamma\pi}{2i}\left[\varphi^{*}(-1+0)-\varphi^{*}(-1-0)\right]=0.$$

Thus, $F^{-}(-1-0)=F^{-}(-1+0)=0$, and as a result, $F^{-}(t)$ satisfies Holder condition on L. Similarly to the case $F^{+}(t)$, it is proved that the series

$$\sum_{n=1}^{\infty}a_n^{-}e^{-int},$$

uniformly converges to $F^{-}\left(e^{it}\right)$ on $[-\pi,\pi]$. Then from the boundary condition (7) it follows that the biorthogonal series

$$\sum_{n=0}^{\infty}a_n^{+}e_n^{+}(t)+\sum_{n=1}^{\infty}a_n^{-}e_n^{-}(t),$$

uniformly converges to $\psi(t)$ on $[-\pi,\pi]$.

The lemma is proved.

Using the representation (8), (9) and the expression of the functions $F^{\pm}(z)$ we establish the validity of the following lemma.

Lemma 4. *Let $0 < Re\beta < 1$ and $\psi(\cdot)$ be an arbitrary Holder function on $[-\pi, \pi]$. Then the series*

$$\sum_{n=0}^{\infty} a_n^+ e_n^+(t) + \sum_{n=1}^{\infty} a_n^- e_n^-(t),$$

where

$$a_n^{\pm} = \int_{-\pi}^{\pi} \psi(t)\, h_n^{\pm}(t)\, \rho(t)\, dt,$$

uniformly converges to $\psi(\cdot)$ on every compact $G \subset (-\pi, \pi)$, and if $\left|1+e^{it}\right|^{-Re\beta} \in L_{p,\rho}$ converges to it in $L_{p,\rho}$, and the following inequality holds

$$-1 < \alpha_k < \frac{p}{q},\ k = -\overline{r,r}. \tag{11}$$

Indeed, the first part follows from the fact that in this case the functions $F^{\pm}\left(e^{it}\right)$ are Holder functions on each compact $G \subset (-\pi, \pi)$. And under fulfilling the inequality (11) the system of exponents $\left\{e^{int}\right\}_{n\in Z}$ forms a basis for $L_{p,\rho}$ and from the inclusion $\left|1+e^{it}\right|^{-Re\beta} \in L_{p,\rho}$ follows that $F^{\pm}\left(e^{it}\right) \in L_{p,\rho}$.

The following theorem follows directly from these lemmas.

Theorem 2. *Let $\rho \in L_1$ and the parameter β satisfy one of the following conditions:*

i) $-Re\beta \in \bigcup_{k=0}^{\infty}(k, k+1)$;

ii) $-\beta \in Z_+$;

iii) $\left|1+e^{it}\right|^{-Re\beta} \in L_{p,\rho}$ and the following inequalities hold

$$-1 < \alpha_k < \frac{p}{q}, k = -\overline{r,r}.$$

Then the system (3) *is complete in $L_{p,\rho}$, for $\forall p \geq 1$, if $\rho \in L_1$.*

Indeed, let us consider the case *i*), the case *ii*) proves similarly. Let, for example, $-Re\beta \in (1,2)$, i.e. $-2 < Re\beta < -1 \Rightarrow Re\tilde{\beta} < 0$, where $\tilde{\beta} = \beta + 1$. Consider the system

$$\left\{e^{i\left(n+\frac{\beta}{2}\right)t}; e^{-i\left(n+\frac{\beta}{2}\right)t}\right\}_{n\in N}. \tag{12}$$

Presenting this system in the form of

$$\left\{e^{i\left(n-1+1+\frac{\beta}{2}\right)t}; e^{-i\left(n+\frac{\beta}{2}\right)t}\right\}_{n\in N},$$

and multiplying it by $e^{-i\frac{\beta}{2}t}$, we get the following system

$$\left\{e^{i\left(n+\frac{\tilde{\beta}}{2}\right)t}\right\}_{n\in Z}, \tag{13}$$

where $\tilde{\beta} = \beta + 1$, as a result, as it follows from Lemma 3, the corresponding biorthogonal series of an arbitrary Holder function $f \in C^{\alpha}[-\alpha, \pi] : f(-\pi) = f(\pi) = 0$ uniformly converges to it on $[-\pi, \pi]$. Denote the partial sums of this series by $S_m(f)$, $m \in N$. Consequently

$$\|f - S_m(f)\|_{p,\rho}^p = \int_{-\pi}^{\pi} |f(t) - S_m(f)(t)|^p \rho(t)\, dt \leq$$

$$\leq \int_{-\pi}^{\pi} \rho(t)\, dt \max_{[-\pi,\pi]} |f(t) - S_m(f)(t)|^p \to 0,\ m \to \infty.$$

Since the set of such functions is dense in $L_{p,\rho}$, hence, we obtain the completeness of the system (13), and at the same time of the system (12) in $L_{p,\rho}$. From the completeness of the system (12) follows the completeness of the system (6) in $L_{p,\rho}$. The remaining cases are proved by mathematical induction. Case *iii*) directly follows from the Lemma 4.

5. Basicity in $L_{p,\rho}$

Before we proceed to the study of basicity let us mention the conditions for completeness and minimality of the system (6) in $L_{p,\rho}$. Thus, by combining the results of Theorems 1 and 2 we arrive at the following conclusion.

Theorem 3. *Let the weight $\rho(\cdot)$ and parameter β satisfy the following conditions*

$$-1 < \alpha_k < \frac{p}{q},\ k = -\overline{r, r};$$

$$-1 < \alpha_{\pm r} - pRe\beta < \frac{p}{q}.$$

Then the system (6) *is complete and minimal in $L_{p,\rho}$, $1 < p < +\infty$.*

Now, we turn to the study of basicity of the system (6) in $L_{p,\rho}$. Let all the conditions of Theorem 3 are fulfilled. Let us consider the boundary value problem (7), where $\psi(\cdot)$ is an arbitrary, finite Holder function on $[-\pi, \pi]$. Applying the Sokhotskii-Plemelj formula to expressions $F^+(z)$ and $F^-(z)$, we obtain

$$F^+\left(e^{it}\right) = ie^{-i\frac{\beta}{2}t}\psi(t) + \Phi^+(\psi),$$

$$F^-\left(e^{it}\right) = ie^{+i\frac{\beta}{2}t}\psi(t) + \Phi^-(\psi),$$

where $\Phi^{\pm}(\psi)$ are corresponding Cauchy type integrals, obtained after the application of these formulas

$$\Phi^+(\psi)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-i\frac{\beta}{2}\theta}\psi(\theta)\, d\theta}{\left(1 + e^{i\theta}\right)^{-\beta}\left(1 - e^{i(t-\theta)}\right)}\left(1 + e^{it}\right)^{-\beta},$$

$$\Phi^-(\psi)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-i\frac{\beta}{2}\theta}\psi(\theta)\, d\theta}{\left(1 + e^{i\theta}\right)^{-\beta}\left(1 - e^{i(t-\theta)}\right)}\left(1 + e^{-it}\right)^{-\beta}.$$

Consider the biorthogonal series of function $\psi(\cdot)$ on system (6):

$$\psi(t) \sim \sum_{n=0}^{\infty} a_n^+ e_n^+(t) + \sum_{n=1}^{\infty} a_n^- e_n^-(t),$$

and let

$$S_{n,m}(\psi) = S_n^+(\psi) + S_m^-(\psi), \; n \in Z_+, \; m \in N,$$

be its partial sums, where

$$S_n^+(\psi) = \sum_{k=0}^{n} a_k^+ e_k^+(t), S_m^-(\psi) = \sum_{k=1}^{m} a_k^- e_k^-(t),$$

and

$$a_k^{\pm} = \int_{-\pi}^{\pi} \psi(t) h_k^{\pm}(t) \rho(t) dt,$$

be corresponding biorthogonal coefficients.

As it has been proven, $S_n^{\pm}$ are the partial sums of biorthogonal series of functions $F^{\pm}(e^{it})$, respectively. From the conditions of the theorem and representation of function $F^{\pm}(\cdot)$ it follows that $F^{\pm} \in L_{p,\rho}$. So, under the conditions (11), the system of exponents $\{e^{int}\}_{n\in Z}$ forms a basis for $L_{p,\rho}$ (see e.g. [31]), then it is clear that $\exists M_1 > 0$:

$$\left\|S_n^{\pm}(\psi)\right\|_{p,\rho} \le M_1 \left\|F^{\pm}(\cdot)\right\|_{p,\rho}, \; \forall n.$$

Consequently

$$\|S_{n,m}(\psi)\|_{p,\rho} \le M_1 \left(\left\|F^+\right\|_{p,\rho} + \left\|F^-\right\|_{p,\rho}\right), \; \forall n, m.$$

We have

$$\left\|F^{\pm}\right\|_{p,\rho} \le \|\psi\|_{p,\rho} + \left\|\Phi^{\pm}(\psi)\right\|_{p,\rho}.$$

On the other hand, from the boundedness of a singular integral operator in weighted space (see e.g. [7, 8]) it follows that under the conditions of the theorem, the following inequality holds

$$\exists M_2 > 0 : \left\|\Phi^{\pm}(\psi)\right\|_{p,\rho} \le M_2 \|\psi\|_{p,\rho}.$$

Taking into account these inequalities as a result we get

$$\left\|S_{n,m}(\psi)\right\|_{p,\rho} \le M \|\psi\|_{p,\rho}, \; \forall n, m. \tag{14}$$

Further, should pay attention to the fact that the linear manifold of finite, Holder functions on $[-\pi, \pi]$ are dense in $L_{p,\rho}$. Then, the uniformly boundedness of projectors $\{S_{n;m}\}$ in $L_{p,\rho}$ directly follows from (14). As a result, taking into account that under the conditions of the theorem the system (6) is complete and minimal in $L_{p,\rho}$, from the basicity criterion we obtain its basicity in $L_{p,\rho}$.

Theorem is proved.

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Sabina R. Sadigova
Institute of Mathematics and Mechanics, The Ministry of Science and Education of Azerbaijan, 9, B.Vahabzadeh Str., AZ1141, Baku, Azerbaijan
E-mail: s_sadigova@mail.ru

Asmar I. Mirzabalayeva
Institute of Mathematics and Mechanics, The Ministry of Science and Education of Azerbaijan, 9, B.Vahabzadeh Str., AZ1141, Baku, Azerbaijan
E-mail: esmer26@mail.ru

Mahbuba H. Orujova
Glodemia Academy, Baku, Azerbaijan
E-mail: orucovamahbubaa@gmail.com

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