Caspian Journal of Applied Mathematics, Ecology and Economics
V. 12, No 2, 2024, December
ISSN 1560-4055

On a Initial-Boundary Value Problem for Fourth-Order Partial Differential Equations with Non-Classical Boundary Conditions

Maryam H. Jafarova

Abstract. In this paper we study some initial-boundary value problem for partial differential equation of fourth order subject the nonclassical boundary conditions. We show the existence, uniqueness and stability of the classical solution of this problem.

Key Words and Phrases: initial-boundary value problem, classical solution, nonclassical boundary conditions, Fourier method

2010 Mathematics Subject Classifications: 35G15, 35L25

1. Introduction

Let $T \in \mathbb{R}$ be the positive constant and $D_T = \{(x,t) \in \mathbb{R}^2 : 0 < x < 1,\, 0 < t < T\}$.

We consider the following initial-boundary value problem for partial differential equation

$$(p(x)u_{x,x}(x,t))_{x,x} - (q(x)u_x(x,t))_x + r(x)u_{tt}(x,t) = f(x,t),\ (x,t) \in D_T, \tag{1}$$

subject the non-local conditions

$$u(x,0) + \delta_1 u(x,T) = \varphi(x),\ \ u_t(x,0) + \delta_2 u(x,T) = \psi(x),\ \ 0 \le x \le 1, \tag{2}$$

and non-classical boundary conditions

$$u(0,t) = \mu_1(t),\ \ 0 \le t \le T, \tag{3}$$

$$u_{xx}(0,t) = \mu_2(t),\ \ 0 \le t \le T, \tag{4}$$

$$p(1)u_{xx}(1,t) + u_x(1,t) = \mu_3(t),\ \ 0 \le t \le T, \tag{5}$$

$$(p(x)u_{xx}(x,t))_x|_{x=1} - q(1)\,u_x(1,t) - r(1)u_{tt}(1,t) = \mu_4(t),\ \ 0 \le t \le T, \tag{6}$$

where δ_i, $i = 1, 2$, are nonnegative constants, $p \in C^2([0,1];(0,+\infty))$, $q \in C^1([0,1];(0,+\infty))$, $r \in C([0,1];(0,+\infty))$, $\mu_i \in C^1([0,T];\mathbb{R})$, $i = 1, 2, 3, 4$, $\varphi, \psi \in C^4([0,1];\mathbb{R})$, $f \in C^{0,1}(D_T)$ is a given function and $u(x,t)$ is the desired function. Moreover, the following conditions hold:

$$\mu_1(0) + \delta_1\mu_1(T) = \varphi(0),\ \ \mu_1'(0) + \delta_2\mu_1'(T) = \psi(0),$$

$$\mu_2(0) + \delta_1\mu_2(T) = \varphi''(0),\ \ \mu_2'(0) + \delta_2\mu_2'(T) = \psi''(0),$$

$$\mu_3(0) + \delta_1\mu_3(T) = p\,(1)\varphi''(1) + \varphi'(1),\ \ \mu_3'(0) + \delta_2\mu_3'(T) = p\,(1)\psi''(1) + \psi'(1),$$

$$\mu_4(0) + \delta_1\mu_4(T) = (p\,(x)\varphi''(x))'\,|_{x=1} - q(1)\varphi'(1) - (f(1,0) + \delta_1 f(1,T)) +$$
$$\left((p\,(x)\varphi''(x))'' - (q(x)\varphi'(x))'\,|_{x=1}\right.,$$

$$\mu_4'(0) + \delta_2\mu_4'(T) = (p\,(x)\psi''(x))'\,|_{x=1} - q(1)\psi'(1) + (f_t(1,0) + \ \delta_2 f_t(1,T)) +$$
$$+ \left((p\,(x)\psi''(x))'' - (q(x)\psi'(x))'\,|_{x=1}\right. .$$

Problem (1)-(6) describes the small bending vibrations of a non-homogeneous rod, the left end of which is elastically fixed, and at the right end the mass is concentrated (see, for example, [6, 9]).

For studying the classical solution of boundary value problems and initial-boundary value problems for partial differential equations one of the main methods is the Fourier method. The justification of this method is traditionally based on the uniform convergence of the series representing the formal solution of the problem and the series obtained by its term-by-term differentiation the required number of times (see, for example, [3-7, 9, 10, 12-14]). Uniform convergence of the series representing the formal solution of the problem and obtained from it by term-by-term differentiation is proved using the basic properties of the corresponding spectral problems.

In this work, using the Fourier method, we prove the existence of a classical solution to problem (1)-(6), and also prove the uniqueness and stability of this solution.

2. Uniqueness of the solution of the initial-boundary value problem (1)-(6)

In this section, we prove the uniqueness of the classical solution to the initial-boundary value problem (1)-(6).

Let

$$C^{4,2}(\overline{D}_T) = \left\{u(x,t) : u(x,t) \in C^2(\overline{D}_T),\ u_{xxxx}(x,t) \in C(\overline{D}_T)\right\}.$$

The classical solution of problem (1)-(6) is called the function $u(x,t) \in C^{4,2}(\overline{D}_T)$ satisfying equation (1) in D_T, conditions (2) in $[0,1]$ and conditions (3)-(6) in $[0,T]$ in the usual sense (see, e.g., [9, 10]).

Theorem 1. *Suppose that* $\delta_1^2 + \delta_2^2 < 1$. *Then problem* (1)-(6) *cannot have more than one classical solution, i.e. if this problem has a classical solution* $u(x,t)$*, then it is unique.*

Proof. Suppose that there are two classical solutions $u_1(x,t)$ and $u_2(x,t)$ of problem (1)-(6) and let

$$v(x,t) = u_1(x,t) - u_2(x,t),\ (x,t) \in \overline{D_T}.$$

Obviously, the function $v(x,t)$, satisfies the following homogeneous equation

$$(p(x)v_{xx}(x,t))_{xx} - (q(x)v_x(x,t))_x + r(x)v_{tt}(x,t) = 0,\ (x,t) \in D_T, \tag{7}$$

and conditions

$$v(x,0) + \delta_1 v(x,T) = 0,\ v_t(x,0) + \delta_2 v_t(x,T) = 0,\ 0 \le x \le 1, \tag{8}$$

$$v(0,t) = 0,\ 0 \le t \le T, \tag{9}$$

$$v_{xx}(0,t) = 0,\ 0 \le t \le T, \tag{10}$$

$$p\,(1)u_{xx}(1,t) + u_x(1,t) = 0,\ 0 \le t \le T, \tag{11}$$

$$(p(x)v_{xx}(x,t))_x\,|_{x=1} - q(1)v_x(1,t) - r(1)v_{tt}(1,t) = 0,\ 0 \le t \le T. \tag{12}$$

Multiplying (7) by the function $2v_t(x,t)$ and integrating the resulting equality in the range from 0 to 1, we obtain

$$\begin{gathered} 2\int\limits_0^1 (p\,(x)v_{xx}(x,t))_{xx}v_t(x,t)dx - 2\int\limits_0^1 (q(x)v_x(x,t))_x v_t(x,t)dx + \\ 2\int\limits_0^1 r(x)v_{tt}(x,t)v_t(x,t)dx = 0. \end{gathered} \tag{13}$$

Note that

$$2\int_0^1 r(x)v_{tt}(x,t)v_t(x,t)dx = \frac{d}{dt}\int_0^1 r(x)v_t^2(x,t)dx,\ 0 \le t \le T. \tag{14}$$

Using the formula for the integration by parts and taking into account conditions (9)-(12) we get the following relations

$$\begin{gathered} 2\int\limits_0^1 (p\,(x)v_{xx}(x,t))_{xx}v_t(x,t)dx = 2(p(x)v_{xx}(x,t))_x\,|_{x=1}\, v_t(1,t) - \\ 2(p\,(x)v_{xx}(x,t))_x\,|_{x=0}\, v_t(0,t) - 2\int\limits_0^1 (p\,(x)v_{xx}(x,t))_x v_{tx}(x,t)dx = \\ 2(p\,(x)v_{xx}(x,t))_x\,|_{x=1}\, v_t(1,t) - 2\int\limits_0^1 (p\,(x)v_{xx}(x,t))_x v_{tx}(x,t)dx = \\ 2(p\,(x)v_{xx}(x,t))_x\,|_{x=1}\, v_t(1,t) - 2p\,(1)v_{xx}(1,t)v_{tx}(x,1) + \ 2p\,(0)v_{xx}(0,t)v_{tx}(0,t) + \\ 2\int\limits_0^1 p(x)v_{xx}(x,t)v_{txx}(x,t)dx = 2(p\,(x)v_{xx}(x,t))_x\,|_{x=1}\, v_t(1,t) - \\ 2p\,(1)v_{xx}(1,t)v_{tx}(x,1) + \tfrac{d}{dt}\int\limits_0^1 p\,(x)v_{xx}^2(x,t)dx, 0 \le t \le T. \end{gathered} \tag{15}$$

$$2\int_0^1 (q(x)v_x(x,t))_x v_t(x,t)dx = 2q(1)v_x(1,t)v_t(1,t) - 2q(0)v_x(0,t)v_t(0,t) - \\ -2\int_0^1 q(x)v_x(x,t)v_{tx}(x,t)dx = 2q(1)v_x(1,t)v_t(1,t) - \frac{d}{dt}\int_0^1 q(x)v_x^2(x,t)dx, \tag{16}$$

Then by (14)-(16) it follows from (13) that

$$\frac{d}{dt}\int_0^1 p(x)v_{xx}^2(x,t)dx + \frac{d}{dt}\int_0^1 q(x)v_x^2(x,t)dx + \frac{d}{dt}\int_0^1 r(x)v_t^2(x,t)dx - \\ 2q(1)v_{xx}(1,t)v_{tx}(1,t) + 2((p(x)v_{xx}(x,t))_x - q(x)v_x(x,t))|_{x=1} v_t(1,t) = 0,$$

$$\frac{d}{dt}\int_0^1 p(x)v_{xx}^2(x,t)dx + \frac{d}{dt}\int_0^1 q(x)v_x^2(x,t)dx + \frac{d}{dt}\int_0^1 r(x)v_t^2(x,t)dx + \\ 2v_x(1,t)v_{tx}(1,t) + 2r(1)v_{tt}(1,t)v_t(1,t) = 0,\ 0 \le t \le T.$$

which implies that

$$\frac{d}{dt}\left(\int_0^1 \left(p(x)v_{xx}^2(x,t) + q(x)v_x^2(x,t) + r(x)v_t^2(x,t)\right)dx + v_x^2(1,t) + r(1)v_t^2(1,t)\right) = 0. \tag{17}$$

Let

$$z(t) = \int_0^1 \left(p(x)v_{xx}^2(x,t) + q(x)v_x^2(x,t) + r(x)v_t^2(x,t)\right)dx + \\ v_x^2(1,t) + r(1)v_t^2(1,t),\ 0 \le t \le T. \tag{18}$$

then it follows from (17) that

$$z'(t) = 0,\ t \in [0,T],$$

and consequently,

$$z(t) = C,\ t \in [0,T] \tag{19}$$

where C is some positive constant.

By (8) we get

$$z(0) - (\delta_1^2 + \delta_2^2)z(T) = \int_0^1 r(x)(v_t^2(x,0) - (\delta_1^2 + \delta_2^2)v_t^2(x,T))dx + \\ \int_0^1 q(x)(v_x^2(x,0) - (\delta_1^2 + \delta_2^2)v_x^2(x,T))dx + \int_0^1 p(x)(v_{xx}^2(x,0) - (\delta_1^2 + \delta_2^2)v_{xx}^2(x,T))dx + \\ r(1)(v_t^2(1,0) - (\delta_1^2 + \delta_2^2)v_t^2(1,T)) + v_x^2(1,0) - (\delta_1^2 + \delta_2^2)v_x^2(1,T) = \\ -\delta_1^2\int_0^1 r(x)v_t^2(x,T))dx - \delta_1^2\int_0^1 q(x)v_x^2(x,T))dx - \delta_2^2\int_0^1 p(x)v_{xx}^2(x,T))dx - \\ - r(1)\delta_1^2 v_t^2(1,T) - \delta_2^2 v_x^2(1,T) = C(1 - (\delta_1^2 + \delta_2^2)) \le 0.$$

whence, by relations $\delta_1^2+\delta_2^2<1$ and $C\geq 0$, implies that $C=0$. Then in view of (19), by (18), we obtain

$$\int\limits_0^1 \left(p(x)v_{xx}^2(x,t)+q(x)v_x^2(x,t)+r(x)v_t^2(x,t)\right)dx+v_x^2(1,t)+r(1)v_t^2(1,t)\equiv 0.$$

Therefore, it follows from last relation that

$$v_t(x,t)\equiv 0,\ v_x(x,t)\equiv 0,\ v_{xx}(x,t)\equiv 0,$$

and consequently,

$$v_t(x,t)=B,\ (x,t)\in\overline{D_T},$$

where B is some constant.

In view of (8) we have

$$v(x,0)+\delta_1 v(x,T)=B(1+\delta_1)=0,$$

which, by $\delta_1\geq 0$, we get $B=0$, i.e.,

$$v(x,t)\equiv 0 \text{ in } \overline{D_T}.$$

The proof of this theorem is complete.

3. Stability of the solution of the initial-boundary value problem (1)-(6)

In this section we prove the stability of the classical solution of the initial-boundary value problem (1)-(6).

Theorem 2. *Let $\delta_1=\delta_2=0$, $\mu_i\equiv 0$, $i=1,2,3,4$, and let the function $u(x,t)\in C^{4,2}(\overline{D_T})$ solves problem* (1)-(6)*. Then for this function the following inequality holds*

$$\begin{aligned}&\int\limits_0^1 \left(p(x)u_{xx}^2(x,t)+q(x)u_x^2(x,t)+r(x)u_t^2(x,t)\right)dx+u_x^2(1,t)+r(1)u_t^2(1,t)\leq\\ &\leq e^{r_0T}\left\{\int\limits_0^1\left(r(x)\psi^2(x)+q(x)[\varphi'(x)]^2+p(x)[\varphi''(x)]^2\right)dx+[\varphi'(1)]^2+\right.\\ &\left.\qquad+r(1)\psi^2(1)+\int\limits_0^T\int\limits_0^1 f^2(x,t)dtdx\right\}.\end{aligned} \tag{20}$$

Proof. Multiplying both parts of (1) by the function $2u_t(x,t)$ and integrating the resulting equality by x in the range from 0 to 1, we obtain

$$\begin{aligned}&2\int\limits_0^1 (p(x)u_{xx}(x,t))_{xx}u_t(x,t)dx-2\int\limits_0^1 (q(x)u_x(x,t))_x u_t(x,t)dx+\\ &2\int\limits_0^1 r(x)u_{tt}(x,t)u_t(x,t)dx=2\int\limits_0^1 f(x,t)u_t(x,t)dx\end{aligned} \tag{21}$$

It is obvious that

$$2\int_0^1 f(x,t)u_t(x,t)dx \le \int_0^1 f^2(x,t)dx + \int_0^1 u_t^2(x,t)dx. \quad (22)$$

By (17) and (22) we get

$$\begin{gathered}\frac{d}{dt}\left(\int_0^1 \left(p\,(x)u_{xx}^2(x,t)+q(x)u_x^2(x,t)+r(x)u_t^2(x,t)\right)dx+u_x^2(1,t)+r(1)u_t^2(1,t)\right)\le\\ \int_0^1 f^2(x,t)dx+\int_0^1 u_t^2(x,t)dx\le\\ \int_0^1 f^2(x,t)dx+\int_0^1 \frac{1}{r(x)}\left(p\,(x)u_{xx}^2(x,t)+q(x)u_x^2(x,t)+r(x)u_t^2(x,t)\right)dx\le\\ \int_0^1 f^2(x,t)dx+\frac{1}{r_0}\int_0^1 \left(p\,(x)u_{xx}^2(x,t)+q(x)u_x^2(x,t)+r(x)u_t^2(x,t)\right)dx,\;\; t\in[0,T],\end{gathered} \quad (23)$$

where $r_0 = \min\limits_{x\in[0,1]} r(x)$.

In view of (18), by (23) we obtain

$$z'(t) \le \int_0^1 f^2(x,t)dx + r_0 z(t),\;\; t\in[0,T],$$

or

$$\frac{d}{dt}\left(z(t)e^{-r_0 t}\right) \le e^{-r_0 t}\int_0^1 f^2(x,t)dx,\; t\in[0,T].$$

It follows from last relation that

$$z(t) \le e^{r_0 T}\left\{z(0) + \int_0^T\int_0^1 f^2(x,t)dxdt\right\},\;\; t\in[0,T]. \quad (24)$$

By initial conditions (2) we have the following relation

$$\begin{gathered}z(0)=\int_0^1 \left(p\,(x)u_{xx}^2(x,0)+q(x)u_x^2(x,0)+r(x)u_t^2(x,0)\right)dx+u_x^2(1,0)+\\ r(1)u_t^2(1,0)=\int_0^1 \left(p(x)\varphi''^2(x)+q(x)\varphi'^2(x)+r(x)\psi^2(x)\right)dx+\varphi'^2(1)+\\ r(1)\psi^2(1)+\int_0^T\int_0^1 f^2(x,t)dtdx\end{gathered} \quad (25)$$

Using (25) from (24) we obtain (20). The proof of this theorem is complete.

Corollary 1. *Let $q(x) > 0$ for $x \in [0,1]$ and let the conditions of Theorem 2 be satisfied. Then the following inequality holds:*

$$|u(x,t)|^2 \leq M \left\{ \int_0^1 \left(p(x)\varphi''^2(x) + q(x)\varphi'^2(x) + r(x)\psi^2(x) \right) dx + \varphi'^2(1) + \right.$$
$$\left. + r(1)\psi^2(1) + \int_0^T \int_0^1 f^2(x,t)dtdx \right\}, \ (x,t) \in \overline{D_T},$$

where $M = e^{r_0 T} \left(\int_0^1 \frac{dx}{q(x)} \right)^{\frac{1}{2}}$.

Remark 1. If the function q takes zero values, then we have the following inequality:

$$|u_x(x,t)|^2 \leq \tilde{D} \left\{ \int_0^1 \left(p(x)\varphi''^2(x) + q(x)\varphi'^2(x) + r(x)\psi^2(x) \right) dx + \varphi'^2(1) + \right.$$
$$\left. r(1)\psi^2(1) + \int_0^T \int_0^1 f^2(x,t)dtdx \right\},$$

4. The existence of a classical solution to problem (1)-(6)

Suppose that $f \equiv 0$ in $\overline{D}_T$ and $\mu_i \equiv 0$ in $[0,T]$ for $i = 1, 2, 3, 4$. In order to solve problem (1)-(6) we apply the method of separation of variables. We will sought for a non-trivial particular solution of equation (1) that satisfies the boundary conditions (3)-(6) in the following form

$$u(x,t) = y(x)\vartheta(t), \ x \in [0,1], \ t \in [0,T]. \tag{26}$$

Taking (26) into account from (1) we obtain

$$(p(x)y''(x))''\vartheta(t) - (q(x)y'(x))'\vartheta(t) + r(x)y(x)\vartheta''(t) = 0 \tag{27}$$

which implies that

$$\frac{(p(x)y''(x))'' - (q(x)y'(x))'}{r(x)y(x)} = -\frac{\vartheta''(t)}{\vartheta(t)} = \lambda, \ \lambda \in \mathbb{C}. \tag{28}$$

Then the functions $y(x)$ and $\vartheta(t)$ will satisfy the following ordinary differential equations

$$(p(x)y''(x))'' - (q(x)y'(x))' = \lambda r(x)y(x), \ 0 < x < 1, \tag{29}$$

and

$$\vartheta''(t) + \lambda\vartheta(t) = 0, \ 0 < t < T, \tag{30}$$

respectively.

By (26) and (28) it follows from (3)-(6) (with the use of conditions $\mu_i \equiv 0$ in $[0,T]$ for $i = 1, 2, 3, 4$) that

$$y(0) = 0, \ y''(0) = 0, \ p(1)y''(1) + y'(1) = 0, \ \mathcal{T}y(1) + \lambda r(1)y(1) = 0,$$

where

$$\mathcal{T}y \equiv (py'')' - qy'.$$

Thus, problem (1), (3)-(6) is reduced by the change of variables (26) to the spectral problem

$$(p\,(x)y''(x))'' - (q(x)y'(x))' = \lambda r(x)y(x),\ 0 < x < 1, \tag{31}$$

$$y(0) = y''(0) = y'(1) + p\,(1)y''(1) = 0, \tag{32}$$

$$\mathcal{T}y(1) + \lambda r(1)y(1) = 0. \tag{33}$$

A more general form of the spectral problem (31)-(33) was considered in [8] (see also [1]), where the oscillatory properties of eigenfunctions and the basis properties of subsystems of eigenfunctions in the space $L_p\,(0,1)$, $1 < p < \infty$, were considered.

Remark 2. By [8, Lemma 2.2 and Theorem 2.2] the eigenvalues of problem (31)-(33) are real and simple and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$. Moreover, multiplying both parts of (31) by y and integrating the resulting relation in the range from 0 to 1 (using integration by parts) and taking the boundary conditions (32), (33) into account we obtain

$$\int\limits_0^1 \left\{p\,(x)y''^2(x) + q(x)y'^2(x)\right\} dx + \frac{1}{p\,(1)}\, y'^2(1) = \lambda \left\{ \int\limits_0^1 r(x)y^2(x)dx + r(1)y^2(1) \right\}$$

whence, by the first condition in (32), implies that the eigenvalues of problem (31)-(33) are positive, i.e., $\lambda_k > 0$ for any $k \in \mathbb{N}$.

Remark 3. It follows from [8, formulas (3.3) and (3.4)] that

$$\sqrt[4]{\lambda_k} = \frac{(k-1)\pi}{\gamma} + O\left(\frac{1}{k}\right), \tag{34}$$

$$\begin{gathered} y_k(x) = \sin \tfrac{(k-1)\,\pi x}{\gamma} - \cos \tfrac{(k-1)\,\pi x}{\gamma} - e^{-\frac{(k-1)\,\pi x}{\gamma}} + \\ (-1)^k e^{\frac{(k-1)\,\pi\,(x-1)}{\gamma}} + O\left(\tfrac{1}{k}\right), \end{gathered} \tag{35}$$

where relation (35) holds uniformly for $x \in [0,1]$ and

$$\gamma = \int\limits_0^1 \left(\frac{r(x)}{p\,(x)}\right)^{1/4} dx.$$

Remark 4. Let s be an arbitrary fixed natural number. Then, by [8, Theorem 5.1], the system $\{y_k\}_{k=1,\,k\neq s}^{\infty}$ of eigenfunctions of problem (31)-(33) forms a basis in the space $L_p\,((0,1);r)$, $1 < p < \infty$, which is an unconditional basis in $L_2((0,1);r)$. Moreover, it follows from the proof of [8, formula (4.3)] that each element v_k of the system $\{v_k\}_{k=1,\,k\neq s}^{\infty}$ conjugate to the system $\{y_k\}_{k=1,\,k\neq s}^{\infty}$ is defined as follows:

$$v_k(x) = \delta_k^{-1} \left\{ y_k - \frac{y_k(1)}{y_s(1)}\, y_s(x) \right\}, \tag{36}$$

where

$$\delta_k = \int_0^1 r(x) y_k^2(x) dx + r(1) y_k^2(1) > 0.$$

Remark 5. In view of [8, Lemma 4.1 and relations (4.11)] we have the following relation

$$v_k(x) = y_k(x) + O\left(\frac{1}{k}\right). \tag{37}$$

$$||y_k||_{2,r}^2 = 1 + O\left(\frac{1}{k}\right) \text{ and } y_k(1) = O\left(\frac{1}{k}\right), \tag{38}$$

where $||\cdot||_{2,r}$ is the norm in $L_2((0,1);r)$.

Let $H = L_2((0,1);r) \oplus \mathbb{C}$ be the Hilbert space with inner product

$$(\hat{u}, \hat{v})_H = (\{y, m\}, \{v, n\})_H = \int_0^1 r(x) y(x) \overline{v(x)}\, dx + \; r(1)^{-1} m \bar{s}\,, \tag{39}$$

We define the linear operator $\mathcal{L} : D(\mathcal{L}) \subset H \to H$ as follows:

$$\mathcal{L}\hat{y} = \mathcal{L}\{y, m\} = \left\{ \frac{1}{r(x)} \left(\mathcal{T} y(x)\right)', -\mathcal{T} y(1) \right\},$$

where

$$D(L) = \{\{y\ (x),\ m\}\ :\ y \in W_2^4(0,1),\ \tfrac{1}{r(x)} \left(\mathcal{T} y(x)\right)' \in L_2(0,1),$$
$$y(0) = y''(0) = y'(0) + p\,(1) y''(1) = 0,\ m = r(1) y(1)\}$$

which is everywhere in H (see [1]). Then problem (31)-(33) is equivalent to the spectral problem

$$L\hat{y} = \lambda \hat{y},\ \hat{y} \in D(L), \tag{40}$$

i.e., the eigenvalues λ_k, $k \in \mathbb{N}$, of problems (31)-(33) and (39) coincide (counting multiplicities), and there exists a one-to-one correspondence between the their eigenfunctions,

$$y_k(x) \leftrightarrow \{y_k(x),\ m_k\},\ \ m_k = r(1) y_k(1).$$

Since r is positive on $[0,1]$, the operator L is a self-adjoint discrete lower-semibounded in H and hence the system of eigenvectors $\{\hat{y}_k\}_{k=1}^{\infty}$ of this operator forms an orthogonal basis in H (see [1]).

For any $k, n \in \mathbb{N}$, $k \neq n$, we have

$$(\hat{y}_k, \hat{y}_n) = 0,$$

and consequently,

$$\int_0^1 r(x) y_k(x) y_n(x) dx + r(1) y_k(1) y_n(1) = 0 \text{ for any } k, \in \mathbb{N},\ k \neq n. \tag{41}$$

Note that $y_k(1) \neq 0$ for any $k \in \mathbb{N}$. Indeed, if $y_k(1) = 0$ for some $k \in \mathbb{N}$, then it follows from (33) that $Ty_k(1) = 0$. Moreover, due to the third condition in (32) we have $y'(1)y''(1) < 0$. Then by the second part of Lemma 2.1 of [2] we get $y'(0)y''(0) < 0$ in contradiction the second condition in (32).

Let k_0 be the arbitrary fixed positive integer. Then by (41) we have

$$\int_0^1 r(x)y_k(x)y_{k_0}(x)dx + r(1)y_k(1)y_{k_0}(1) = 0 \text{ for any } k \in \mathbb{N},\ k \neq k_0,$$

which implies that

$$r(1)y_k(1) + \frac{\ }{y_{k_0}(1)} \int_0^1 r(x)y_k(x)y_{k_0}(x)dx = 0 \text{ for any } k \in \mathbb{N},\ k \neq k_0, \tag{42}$$

Thus, by (42), λ_k, $k \in \mathbb{N}$, $k \neq k_0$, are eigenvalues and y_k, $k \in \mathbb{N}$, $k \neq k_0$, are corresponding eigenfunctions of the following spectral problem

$$\begin{cases} (p\,(x)y''(x))'' - (q(x)y'(x))' = \lambda r(x)y(x),\ 0 < x < 1, \\ y(0) = y''(0) = y'(1) + p\,(1)y''(1) = 0, \\ r(1)y(1) + \frac{1}{y_{k_0}(1)} \int_0^1 r(x)y(x)y_{k_0}(x)dx = 0. \end{cases} \tag{43}$$

Note that, unlike problem (31)-(33), problem (43) does not contain a spectral parameter in the boundary conditions.

By first relation of (38), without loss of generality, we can assume that the functions y_k, $k \in \mathbb{N}$, are normalized in $L_2((0,1);r)$. Then, by Remark 4.3, the system $\{y_k(x)\}_{k=1,\,k\neq k_0}^{\infty}$ forms a Riesz basis in the space $L_2((0,1);r)$. In this case the system $\{v_k(x)\}_{k=1,\,k\neq k_0}^{\infty}$, where

$$v_k(x) = \delta_k^{-1}\left\{ y_k - \frac{y_k(1)}{y_{k_0}(1)}\, y_{k_0}(x)\right\},$$

is conjugate to the system $\{y_k(x)\}_{k=1,\,k\neq k_0}^{\infty}$. Hence for any function $g \in L_2((0,1);r)$ we have

$$g = \sum_{k=1, k\neq k_0}^{\infty} g_k y_k(x), \tag{44}$$

where

$$\begin{aligned} g_k = \int_0^1 r(x)g(x)v_k(x)dx = \delta_k^{-1} \int_0^1 r(x)g(x)y_k(x)dx \\ - \delta_k^{-1} \frac{y_k(1)}{y_{k_0}(1)} \int_0^1 r(x)g(x)y_{k_0}(x)dx. \end{aligned} \tag{45}$$

Let the following conditions hold:

$$g(x),\ g'(x),\ g''(x),\ \mathcal{T}g(x) \in C[0,1],\ g(0)=0,\ g''(0)=0,\ g'(1)+p(1)g''(1)=0,$$

$$J(g) = r(1)g(1) + \frac{1}{y_{k_0}(1)} \int_0^1 r(x)g(x)y_{k_0}(x)dx = 0 \text{ and } \frac{1}{r(x)}(\mathcal{T}g(x))' \in L_2(0,1).$$

For any $g \in D(\mathcal{L})$ we have

$$(L\hat{y}_k, \overline{\hat{g}}) = \lambda_k(\hat{y}_k, \overline{\hat{g}}),\ k \in \mathbb{N},$$

whence, by (39), we get

$$\lambda_k \int_0^1 r(x)y_k(x)g(x)dx + \lambda_k r(1)y_k(1)g(1) = \lambda_k(\hat{y}_k, \overline{\hat{g}})_H = (L\hat{y}_k, \overline{\hat{g}})_H =$$

$$\left(\hat{y}_k, L\overline{\hat{g}}\right)_H = \int_0^1 y_k(x)(Tg(x))'dx - y_k(1)Tg(1),\ k \in \mathbb{N}.$$

Thus, for any $g \in D(\mathcal{L})$ we obtain

$$\lambda_k \int_0^1 r(x)y_k(x)g(x)dx = -\lambda_k r(1)y_k(1)g(1) - y_k(1)Tg(1) + \int_0^1 y_k(x)(Tg(x))'dx,\ k \in \mathbb{N}.$$

whence implies that

$$\lambda_{k_0}\frac{y_k(1)}{y_{k_0}(1)} \int_0^1 r(x)y_{k_0}(x)g(x)dx = -\lambda_{k_0} r(1)y_k(1)g(1) - y_k(1)Tg(1) +$$

$$\frac{y_k(1)}{y_{k_0}(1)} \int_0^1 r(x)y_{k_0}(x)(Tg(x))'dx,\ k \in \mathbb{N}.$$

It follows from two last relations that

$$\lambda_k \int_0^1 r(x)g(x)\left\{y_k(x) - \frac{y_k(1)}{y_{k_0}(1)}\,y_{k_0}(x)\right\}dx + (\lambda_k - \lambda_{k_0})\frac{y_k(1)}{y_{k_0}(1)} \int_0^1 r(x)y_{k_0}(x)g(x)dx =$$

$$-(\lambda_k - \lambda_{k_0})r(1)y_k(1)g(1) + \int_0^1 \left\{y_k(x) - \frac{y_k(1)}{y_{k_0}(1)}\,y_{k_0}(x)\right\}(\mathcal{T}g)'(x)dx,$$

and consequently,

$$\lambda_k \int_0^1 r(x)g(x)\left\{y_k(x) - \frac{y_k(1)}{y_{k_0}(1)}\, y_{k_0}(x)\right\} dx =$$

$$-(\lambda_k - \lambda_{k_0})y_k(1)\left\{r(1)g(1) + \frac{1}{y_{k_0}(1)}\int_0^1 r(x)y_{k_0}(x)g(x)dx\right\} +$$

$$\int_0^1 \left\{y_k(x) - \frac{y_k(1)}{y_{k_0}(1)}\, y_{k_0}(x)\right\}(\mathcal{T}g)'(x)dx.$$

Since $J(g) = 0$ we have the following relation

$$\int_0^1 r(x)g(x)v_k(x)dx = \frac{1}{\lambda_k}\int_0^1 (\mathcal{T}g)'(x)v_k(x)dx.$$

Lemma 1. *Let the conditions* $g \in C^3[0,1]$, $g \in W_2^4(0,1)$, $g(0) = g''(0) = g'(1) + p\,(1)g''(1) = 0$ *and* $J(g) = 0$ *be satisfied. Then the following relation holds:*

$$g_k = \lambda_k^{-1} g_{k,1},$$

where

$$g_{k,1} = \int_0^1 r(x)G(x)v_k(x)dx,\ G(x) = \frac{(\mathcal{T}g)'(x)}{r(x)},\ x \in [0,1].$$

Corollary 2. Let the conditions of Lemma 4.1 be satisfied. Then one has the relation

$$\sum_{k=1,\,k\neq k_0}^{\infty} \lambda_k^2 g_k^2 = \sum_{k=1,k\neq k_0}^{\infty} g_{k,1}^2 \leq \int_0^1 \frac{(Tg(x))'^2}{r(x)}\, dx\,.$$

Lemma 2. *Let* $g_1 = (Tg)'$ *and the following conditions hold:* $p \in C^4[0,1]$, $q \in C^2[0,1]$, $g \in C^7[0,1]$, $g \in W_2^8(0,1)$, $g(0) = g''(0) = g'(1) + p\,(1)g''(1) = 0$, $J(g) = 0$ *and* $g_1(0) = g_1''(0) = g_1'(1) + p\,(1)g_1''(1) = 0$, $J(g_1) = 0$. *Then we have the following relation:*

$$g_k = \lambda_k^{-2} g_{k,2},\ k \in \mathbb{N},\ k \neq k_0,$$

where

$$g_{k,2} = \int_0^1 r(x)g_1(x)v_k(x)dx.$$

Corollary 3. Let the conditions of Lemma 4.2 hold. Then one has the relation

$$\sum_{k=1,\, k\neq k_0}^{\infty} \lambda_k^4 g_k^2 = \sum_{k=1,k\neq k_0}^{\infty} g_{k,2}^2 \leq \int_0^1 \frac{(Tg(x))'^2}{r(x)}\, dx\,.$$

We will seek the solution to problem (1)-(6) in the form

$$u(x,t) = \sum_{k=1,\, k\neq k_0}^{\infty} u_k(t) y_k(x), \tag{46}$$

where

$$u_k(t) = \int_0^1 r(x)u(x,t)v_k(x)dx,$$

$$v_k(x) = \delta_k^{-1}\left(y_k(x) - \frac{y_k(1)}{y_{k_0}(1)}\, y_{k_0}(x)\right),\ k \in \mathbb{N},\ k \neq k_0.$$

We apply the method of separation of variables to determine the desired functions $u_k(t)$, $k \in \mathbb{N}$, $k \neq k_0$. Then from (1) we obtain

$$u_k''(t) + \lambda_k u_k(t) = 0,\ k \in \mathbb{N},\ k \neq k_0,\ t \in [0,T], \tag{47}$$

$$u_k(0) + \delta_1 u_k(T) = \varphi_k,\ u_k'(0) + \delta_2 u_k'(T) = \psi_k,\ k \in \mathbb{N},\ k \neq k_0, \tag{48}$$

where

$$\varphi_k = \int_0^1 r(x)\varphi(x)v_k(x)dx,\ \psi_k = \int_0^1 r(x)\psi(x)v_k(x)dx,\ k \in \mathbb{N},\ k \neq k_0.$$

Solving problem (47), (48) by using Remark 4.1 we get

$$u_k(t) = \frac{1}{\varrho_k(T)}\left[\varphi_k(\cos\rho_k t + \delta_2 \cos\rho_k(T-t)) + \frac{\psi_k}{\rho_k}(\sin\rho_k t - \delta_1 \sin\rho_k(T-t))\right],$$

where

$$\rho_k = \sqrt{\lambda_k},\ \ \varrho_k(T) = 1 + (\delta_1+\delta_2)\cos\rho_k T + \delta_1\delta_2.$$

The following theorem is the main result of this paper.

Theorem 3. *Let the following conditions hold:*

(i) $1+\delta_1\delta_2 \geq \delta_1+\delta_2$,

(ii) $\mu_i \equiv 0$, $i = 1, 2, 3, 4$, $p \in C^4([0,1];(0,+\infty))$, $q \in C^2([0,1];[0,+\infty))$,

(iii) $\varphi \in C^7([0,1];\mathbb{R})$, $\phi \in W_2^8(0,1)$, $\varphi(0) = \varphi''(0) = \varphi'(1) + p(1)\varphi''(1) = 0$, $J(\varphi) = 0$

and $\phi(0) = \phi''(0) = \phi'(1) + p\,(1)\phi''(1) = 0,\ J(\phi) = 0,$ *where* $\phi = \frac{1}{r}\,(T\varphi)'$,

(iv) $\psi \in C^3([0,1];\mathbb{R}),\ \psi \in W_2^4(0,1),\ \psi''(0) = \psi'(1) + p\,(1)\psi''(1) = 0.$
Then the function

$$u(x,t) = \sum_{k=1,\,k\neq k_0}^{\infty} \frac{1}{\varrho_k(T)}\left[\varphi_k(\cos\rho_k t + \delta_2\cos\rho_k(T-t)) + \right.$$

$$\left. \frac{\psi_k}{\rho_k}(\sin\rho_k t - \delta_1 \sin\rho_k(T-t))\right] y_k(x)$$

is a classical solution of problem (1)-(6).

The proof of this theorem is similar to the proof of the justification of the Fourier method in [10, § 23.5] (see also [9]) with the use of Lemmas 1, 2 and Corollaries 2, 3.

References

[1] Z.S. Aliev, Basis Properties in of systems of root functions of a spectral problem with spectral parameter in a boundary condition. *Diff. Equ.* **47**(6) (2011) 766-777.

[2] D.O. Banks, G.J. Kurowski, A Prüfer transformation for the equation of a vibrating beam subject to axial forces. *J. Differential Equations* **24**(1) (1977), 57-74.

[3] B.B. Bolotin, *Vibrations in Technique: Handbook in 6 Volumes, The Vibrations of Linear Systems, I.* Engineering Industry, Moscow, 1978.

[4] M.Sh. Burlutskaya, A.P. Khromov, Classical solution of a mixed problem with involution, *Dokl. Math.* **82**(3) (2010), 865-868.

[5] M.Sh. Burlutskaya, A.P. Khromov, Fourier method in an initial-boundary value problem for a first-order partial differential equation with involution, *Comput. Math. and Math. Phys.* **51**(12) (2011), 2102-2114

[6] V.A. Chernyatin, To clarify the theorem of existence of the classical solution of the mixed problem for onedimensional wave equation. *Differ. Equ.* **21**(9) (1985), 1569-1576 (in Russian).

[7] V.A. Ilin, The solvability of mixed problems for hyperbolic and parabolic equations. *Rus. Math. Surv.* **15**(1) (1960), 85-142.

[8] N.B. Kerimov, Z.S. Aliev, On the basis property of the system of eigenfunctions of a spectral problem with spectral parameter in a boundary condition, *Differ. Equ.* **43**(7) (2007), 905-915.

[9] O.A. Ladyzhenskaya, *Mixed problem for a hyperbolic equation.* Moscow, Gostekhizdat, 1953 (in Russian).

[10] I.G. Petrovsky, *Lectures on partial differential equations.* Interscience, New York, 1954.

[11] M. Roseau, *Vibrations in Mechanical Systems, Analytical Methods and Applications.* Springer, Berlin, 1987.

[12] Z. Wang Z, T. Hillen, Classical solutions and pattern formation for a volume filling chemotaxis model, *Chaos* **17**(3) (2007), 13 p.

[13] M. Winkler, Classical solutions to Cauchy problems for parabolic-elliptic systems of Keller-Segel type, *Open Math.* **21**(1) (2023), 19 p.

[14] N.V. Zaitseva, Classical solutions of hyperbolic differential-difference equations in a half-space. *Differ. Equ.* **57**(12) (2021), 1629-1639.

Maryam H. Jafarova
Baku State University, Baku, Azerbaijan
E-mail: maryam.bzade20@mail.ru

Received 27 June 2024
Accepted 29 September 2024