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On the Basis Property in L_p of Eigenfunctions of a Differential Operator with Integral Boundary Conditions

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Abstract. This paper studies a second-order differential operator with integral boundary conditions of the form $\int_0^1 \varphi_\nu(x)\, y(x) dx = 0, \quad \nu = 1, 2$. Conditions on the functions $\varphi_\nu(x), \nu = 1, 2$, are found under which the system of eigenfunctions of the differential operator forms a basis in a certain subspace of $L_p(0,1)$, $1 < p < \infty$, of codimension 2. The question of a possible extension of this system to a basis of the entire space $L_p(0,1)$ is also considered.

Key Words and Phrases: second-order differential operator, eigenfunction, boundary conditions, basis.

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1. Introduction and Formulation of Problem

Consider the linear differential expression

$$l(y) = -y'' + q(x)\, y, x \in (0,1), \tag{1}$$

and the boundary conditions

$$U_1(y) = U_2(y) = 0, \tag{2}$$

where $q(x)-$ is a complex-valued integrable function on $[0,1]$ and $U_1(y)$ and $U_2(y)$ - are the corresponding boundary forms. The differential expression (1) and the boundary conditions (2) generate a differential operator L with a domain $\mathrm{D}(L)$ in some functional space X. We are interested in the behavior of the eigenvalues and eigenfunctions of this differential operator. This problem has been well studied in the case of regular boundary conditions $U_\nu(y) = 0$, $\nu = 1, 2$, (see [1,2] and the references therein). The case of irregular, as well as more general regular boundary conditions involving certain integrals of the function $y(x)$ and its derivatives, has been considered in [3-7]. These works studied the spectral properties of the corresponding operator (spectrality, eigenvalues, and eigenfunctions, the adjoint problem), primarily in the space $L_2(0,1)$). Additionally, works [8-13] investigated similar problems in $L_p(0,1)$, and provided abstract approaches for their analysis. However, in most cases, the boundary forms generated an unbounded functional in

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the considered space. In this case, the operator had a dense domain, allowing the construction of an adjoint operator or assuming the regularity of the boundary conditions [1, 2, 4, 8-12]. Here, we will consider integral boundary conditions of the form

$$U_\nu\left(y\right)=\int_0^1 \varphi_\nu\left(x\right) y\left(x\right)dx=0,\ \nu=1,2, \tag{3}$$

where $\varphi_\nu\left(x\right)$ - are given linearly independent functions belonging to the space $L_q\left(0,1\right)$, $\frac{1}{p}+\frac{1}{q}=1$. These conditions are not regular in the sense of Birkhoff [1], and there is no corresponding adjoint operator for them. Similar conditions have been used for other purposes in [6, 7]. In [14], the problem (1),(3) was studied under stricter conditions on the functions $q\left(x\right)$ and $\varphi_\nu\left(x\right)$, where the asymptotics of eigenvalues and eigenfunctions were obtained. In [15], a Riesz basis theorem for the system of eigenfunctions in a certain subspace of $L_2\left(0,1\right)$ was proven. In [16], the completeness and minimality of the eigenfunctions in a certain subspace of $L_p\left(0,1\right)$ were proven. It is worth noting that differential equations with nonlocal integral-type conditions have interesting applications in mechanics [17] and in the theory of diffusion processes [18].

2. Preliminaries

We define the differential operator L in the space $L_p\left(0,1\right)$, $1<p<\infty$, with the domain

$$D(L)=\{y(x)\in W_p^2(0,1);\ l(y)\in L_p(0,1);\ U_1(y)=0,\ U_2(y)=0\},$$

which acts as

$$Ly=l(y),\ \forall y\in D(y).$$

Consider the eigenvalue problem for the operator L:

$$Ly=\ \lambda y. \tag{4}$$

Let us set $\lambda=\rho^2$ and denote by $y_1(x,\rho)$ the solution of equation (4) that satisfies the initial conditions $y(0)=0,\ y'(0)=\rho$, and by $y_2(x,\rho)$– the solution of equation (4) that satisfies the initial conditions $y(0)=1,\ y'(0)=0$. It is obvious that these solutions are linearly independent (i.e., they form a fundamental system of solutions of equation (4)). It is known [2] that these functions are also solutions of the following integral equations:

$$y_1\left(x,\rho\right)=\sin\rho x\ +\frac{1}{\rho}\int_0^x q\left(t\right)y_1\left(t,\rho\right)\sin\rho\left(x-t\right)\,dt, \tag{5}$$

$$y_2\left(x,\rho\right)=\cos\rho x\ +\frac{1}{\rho}\int_0^x q\left(t\right)y_2\left(t,\rho\right)\sin\rho\left(x-t\right)\,dt. \tag{6}$$

These solutions satisfy the estimates as $\mid\rho\mid\to\infty$ (see. [2]):

$$\mid y_1\left(x,\rho\right)\mid\leq Ce^{|Im\rho|x},\ \ \mid y_2\left(x,\rho\right)\mid\leq Ce^{|Im\rho|x}\ , \tag{7}$$

and are therefore bounded in the strip $\mid Im\rho\mid\leq h$, for some $h>0$. For further purposes, we need to refine these estimates.

Lemma 1. *For the functions $y_1(x,\rho)$ and $y_2(x,\rho)$ in the strip $|\, Im\rho \,|\le h$, the following asymptotic formulas hold as $|\,\rho\,|\to\infty$:*

$$y_1(x,\rho)=\sin\rho x-\frac{1}{2\rho}\int_0^x q(t)\,dt\cos\rho x\;+\frac{1}{2\rho}\int_0^x q(t)\cos\rho(x-2t)\;dt+O\left(\frac{1}{\rho^2}\right); \quad (8)$$

$$y_2(x,\rho)=\cos\rho x\;+\frac{1}{2\rho}\int_0^x q(t)\,dt\;\sin\rho x\;+\frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\;dt+O\left(\frac{1}{\rho^2}\right). \quad (9)$$

Proof. Substituting the expressions for the functions $y_1(t,\rho)$, $y_2(t,\rho)$, given by formulas (5) and (6), into the right-hand side of formulas (5) and (6) under the integral signs:

$$y_1(x,\rho)=\sin\rho x\;+\frac{1}{\rho}\int_0^x q(t)\left[\sin\rho t\;+\frac{1}{\rho}\int_0^t q(\tau)\,y_1(\tau,\rho)\sin\rho(t-\tau)\;d\tau\right]\sin\rho(x-t)\;dt=$$

$$=\sin\rho x\;+\frac{1}{\rho}\int_0^x q(t)\sin\rho t\;\;\sin\rho(x-t)\;dt+$$

$$+\frac{1}{\rho^2}\int_0^x q(t)\left(\int_0^t q(\tau)\,y_1(\tau,\rho)\sin\rho(t-\tau)\;d\tau\right)\sin\rho(x-t)\;dt=$$

$$=\sin\rho x\;-\frac{1}{2\rho}\int_0^x q(t)\,dt\cos\rho x\;+\frac{1}{2\rho}\int_0^x q(t)\cos\rho(x-2t)\;dt+\frac{1}{\rho^2}r_1(x,\rho);$$

$$y_2(x,\rho)=\cos\rho x\;+\frac{1}{\rho}\int_0^x q(t)\left[\cos\rho t\;+\frac{1}{\rho}\int_0^t q(\tau)\,y_2(\tau,\rho)\,d\tau\right]\sin\rho(x-t)\;dt=$$

$$=\cos\rho x\;+\frac{1}{\rho}\int_0^x q(t)\cos\rho t\;\sin\rho(x-t)\;dt+$$

$$+\frac{1}{\rho^2}\int_0^x q(t)\left(\int_0^t q(\tau)\,y_2(\tau,\rho)\sin\rho(t-\tau)\;d\tau\right)\sin\rho(x-t)\;dt=$$

$$=\cos\rho x\;+\frac{1}{2\rho}\int_0^x q(t)\,dt\;\sin\rho x\;+\frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\;dt+\frac{1}{\rho^2}r_2(x,\rho),$$

where denoted by

$$r_i(x,\rho)=\int_0^x q(t)\left(\int_0^t q(\tau)\,y_i(\tau,\rho)\sin\rho(t-\tau)\;d\tau\right)\sin\rho(x-t)\;dt,\;\;i=1,2.$$

Considering the known inequalities

$$|\,\sin\rho x\,|\le e^{|Im\rho|x},\;\;|\,\cos\rho x\,|\le e^{|Im\rho|x},\;\;x\in[0,1],$$

As well as inequalities (7), we obtain

$$|\,r_i(x,\rho)\,|\le\int_0^x |q(t)|\left(\int_0^t |q(\tau)|\,|y_i(\tau,\rho)|\,|\sin\rho(t-\tau)\,|\,d\tau\right)|\sin\rho(x-t)\,|\,dt\le$$

$$\leq C\int_0^x |q(t)| \left(\int_0^t |q(\tau)|\, e^{|Im\rho|\tau} e^{|Im\rho|(t-\tau)} d\tau\right) e^{|Im\rho|(x-t)} dt \leq$$

$$\leq Ce^{|Im\rho| x}\int_0^x |q(t)| \left(\int_0^t |q(\tau)|\, d\tau\right) dt \leq Ce^{|Im\rho|}\left(\int_0^1 |q(t)|\, dt\right)^2.$$

Thus, for $|Im\rho| \leq h, \quad |\,\rho\,| \to \infty$, the following estimates hold uniformly for $x \in [0,1]$:

$$|\, r_1(x,\rho)\, | = O(1)\,, |\, r_2(x,\rho)\, | = O(1)\,,$$

From this, the validity of formulas (8) and (9). Lemma is proved.

Let us present briefly the main definitions and facts which will be used in what follows. Let X be a B-space. A system$\{x_n\}_{n\in N}$ of elements X is said to be complete in X if $\overline{L\left(\{x_n\}_{n\in N}\right)} = X$; that is, any element of the space X can be approximated by a linear combination of elements of this system with any accuracy in the norm of the space X.

A system $\{x_n\}_{n\in N}$ of elements X is said to be minimal in X if $x_n \notin \overline{L\left(\{x_k\}_{k\neq n}\right)}$. It is well known that a system $\{x_n\}_{n\in N}$ is minimal if and only if there exists a biorthogonal system which is dual to it, that is, a system of linear functionals $\{x_n^*\}_{n\in N}$ from X^* such that $\langle x_n, x_k^*\rangle = \delta_{nk}$ for all $n,\ k \in N$. Moreover, if the initial system is complete and minimal in X, then the biorthogonal system is uniquely defined.

A system $\{x_n\}_{n\in N}$ forms a basis of the space X if, for any element $x \in X$, there exists a unique expansion into a series

$$x = \sum_{n=1}^{\infty} c_n x_n$$

converging in the norm of the space X.

Two systems $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ of a B-space X are called equivalent if there exists an automorphism that maps one of these systems to the other. A system equivalent to a basis is itself a basis in the same space.

Any basis is a complete and minimal system in X, and, therefore, we can uniquely find its biorthogonal dual system $\{x_n^*\}_{n\in N}$ and hence the expansion of any element $x \in X$ with respect to the basis $\{x\}_{n\in N}$ coincides with its biorthogonal expansion, that is, $c_n = \langle x, x_n^*\rangle$ for all $n \in N$.

We will use also some facts about p - closure bases. Concerning these facts more details one can see the works [19, 20].

Systems $\{x_n\}_{n\in N}, \quad \{y_n\}_{n\in N} \subset X$ in B -space X are called p-closure if

$$\sum_{n=1}^{\infty} \|x_n - y_n\|_X^p < \infty.$$

The minimal system $\{x_n\}_{n\in N} \subset X$ with biorthogonal system $\{x_n^*\}_{n\in N} \subset X^*$ is called p- besselian, if for any $x \in X$

$$\left(\sum_{n=1}^{\infty} |\langle x, x_n^*\rangle|^p\right)^{\frac{1}{p}} \leq M\|x\|_X.$$

If the basis $\{x_n\}_{n\in N}$ for X is p- besselian, then we call it as p-basis.

It is valid the following

Theorem 1. *[20] Let the system $\{x_n\}_{n\in N}$ is p -basis for B- space X and the system $\{y_n\}_{n\in N} \subset X$ is p'– clouser to it, $1 < p < \infty$. Then the following assertions are equivalent:*

1. *$\{y_n\}_{n\in N}$ is complete in X;*
2. *$\{y_n\}_{n\in N}$ is minimal in X;*
3. *$\{y_n\}_{n\in N}$ is isomorphic to $\{x_n\}_{n\in N}$ basis for X.*

3. Asymptotics of Eigenvalues and Eigenfunctions

The main result of this section is:

Theorem 2. *Let the function $q(x)$ be integrable, and k, let the functions $\varphi_\nu(x)$, $\nu = 1,2$, belong to the class $W_1^2(0,1)$. Suppose that*

$$\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0, \tag{10}$$

where we denote $\alpha_\nu = \varphi_\nu(0)$, $\beta_\nu = \varphi_\nu(1)$. Then the eigenvalues of the operator L are asymptotically simple, and the following asymptotic formula holds:

$$\lambda_k = \rho_k^2, \quad \rho_k = \pi k + O\left(\frac{1}{k}\right), \quad k \to \infty.$$

Proof. To find the eigenvalues of the operator L consider the determinant

$$\Delta(\rho) = \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix},$$

where $y_1 = y_1(x,\rho)$, $y_2 = y_2(x,\rho)$ – form a fundamental system of solutions of equation (4). Using formulas (8) and (9), we obtain

$$U_\nu(y_1) = \int_0^1 y_1(x,\rho)\varphi_\nu(x)\,dx = \int_0^1 \varphi_\nu(x)\sin\rho x\,dx - \frac{1}{2\rho}\int_0^1 \varphi_\nu(x)\cos\rho x\int_0^x q(t)\,dtdx +$$

$$+\frac{1}{2\rho}\int_0^1 \varphi_\nu(x)\int_0^x q(t)\cos\rho(x-2t)\,dtdx + O(\frac{1}{\rho^2}) =$$

$$= \frac{1}{\rho}\left(-\varphi_\nu(x)\cos\rho x + \varphi'_\nu(x)\sin\rho x\right)\Big|_0^1 - \frac{1}{\rho^2}\int_0^1 \varphi''_\nu(x)\sin\rho x\,dx -$$

$$+\frac{1}{2\rho}\int_0^1 q(t)\left(\int_t^1 \varphi_\nu(x)\,cos\rho(x-2t)\,dx\right)dt + O\left(\frac{1}{\rho^2}\right) =$$

$$= \frac{1}{\rho}\left(-\beta_\nu\cos\rho + \alpha_\nu\right) + \frac{1}{\rho^2}\varphi'_\nu(1)\sin\rho - \frac{1}{\rho^2}\int_0^1 \varphi"_\nu(x)\sin\rho\,x dx -$$

$$-\frac{1}{2\rho^2}\varphi_\nu(1)\sin\rho\int_0^1 q(t)\,dt+\frac{1}{2\rho^2}\int_0^1 \varphi'_\nu(x)\int_0^x q(t)\,dt\,\sin\rho x\,dx+$$

$$+\frac{1}{2\rho^2}\int_0^1 \varphi_\nu(x)\,q(x)\sin\rho x\,dx+\frac{1}{2\rho^2}\int_0^1 q(t)\left(\varphi_v(x)\sin\rho(x-2t)\,|_{x=t}^{x=1}\right)dt-$$

$$-\frac{1}{2\rho^2}\int_0^1 q(t)\int_t^1 \varphi'_\nu(x)\sin\rho(x-2t)\,dxdt+O\left(\frac{1}{\rho^2}\right)=$$

$$=\frac{1}{\rho}\left(-\beta_\nu\cos\rho+\alpha_\nu\right)+O\left(\frac{1}{\rho^2}\right); \tag{11}$$

$$U_\nu(y_2)=\int_0^1 y_2(x,\rho)\varphi_\nu(x)\,dx=\int_0^1 \varphi_\nu(x)\cos\rho x\,dx+\frac{1}{2\rho}\int_0^1 \varphi_\nu(x)\sin\rho x\int_0^x q(t)\,dtdx+$$

$$+\frac{1}{2\rho}\int_0^1 \varphi_\nu(x)\int_0^x q(t)\sin\rho(x-2t)\,dtdx+O(\frac{1}{\rho^2})=$$

$$=\frac{1}{\rho}\beta_\nu\sin\rho-\frac{1}{\rho^2}\varphi'_\nu(1)\cos\rho-\frac{1}{\rho^2}\int_0^1 \varphi''_\nu(x)\cos\rho x\,dx-$$

$$-\frac{1}{2\rho^2}\left(\varphi_\nu(x)\int_0^x q(t)\,dt\cos\rho x\right)\Bigg|_{x=0}^{x=1}+\frac{1}{2\rho^2}\int_0^1 q(t)\int_t^1 \varphi_v(x)\sin\rho(x-2t)\,dtdx=$$

$$=\frac{1}{\rho}\beta_\nu\sin\rho-\frac{1}{\rho^2}\varphi'_\nu(1)\cos\rho-\frac{1}{\rho^2}\int_0^1 \varphi''_\nu(x)\cos\rho\,xdx-\varphi_\nu(1)\cos\rho\int_0^1 q(t)\,dt-$$

$$-\frac{1}{2\rho^2}\int_0^1 q(t)\left(\varphi_\nu(x)\cos\rho(x-2\mathrm{t})\,|_{x=t}^{x=1}-\int_t^1 \varphi'_\nu(x)\cos\rho(x-2t)\,dx\right)dt+O\left(\frac{1}{\rho^2}\right)=$$

$$=\frac{1}{\rho}\beta_\nu\sin\rho+O\left(\frac{1}{\rho^2}\right). \tag{12}$$

Thus, the following relations are obtained:

$$U_\nu(y_1)=\frac{1}{\rho}\left(-\beta_\nu\cos\rho+\alpha_\nu\right)+O\left(\frac{1}{\rho^2}\right),$$

$$U_\nu(y_2)=\frac{1}{\rho}\beta_\nu\sin\rho+O\left(\frac{1}{\rho^2}\right).$$

Taking these relations into account, we obtain

$$\varDelta(\rho)=\begin{vmatrix}\frac{1}{\rho}\left(-\beta_1\cos\rho+\alpha_1\right)+O\left(\frac{1}{\rho^2}\right) & \frac{1}{\rho}\beta_1\sin\rho+O\left(\frac{1}{\rho^2}\right)\\ \frac{1}{\rho}\left(-\beta_2\cos\rho+\alpha_2\right)+O\left(\frac{1}{\rho^2}\right) & \frac{1}{\rho}\beta_2\sin\rho+O\left(\frac{1}{\rho^2}\right)\end{vmatrix}=$$

$$=\frac{1}{\rho^2}\begin{vmatrix}-\beta_1\cos\rho+\alpha_1+O\left(\frac{1}{\rho}\right) & \beta_1\sin\rho+O\left(\frac{1}{\rho}\right)\\ -\beta_2\cos\rho+\alpha_2+O\left(\frac{1}{\rho}\right) & \beta_2\sin\rho+O\left(\frac{1}{\rho}\right)\end{vmatrix}=$$

$$= \frac{1}{\rho^2}\left(\alpha_1\beta_2 - \alpha_2\beta_1\right)\sin\rho \ + O\left(\frac{1}{\rho^3}\right) = \frac{1}{\rho^2}\left(\left(\alpha_1\beta_2 - \alpha_2\beta_1\right)\sin\rho \ + O\left(\frac{1}{\rho}\right)\right).$$

According to condition (10) $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, applying Rouché's theorem and using the standard method (see [1, p. 77]), we obtain the asymptotics of the zeros of $\Delta(\rho)$:

$$\rho_k = \pi k + O\left(\frac{1}{k}\right), \quad k \to \infty.$$

The theorem is proved.

Remark 1. *The numbering of the zeros will be refined later. For now, we can only assert that they are asymptotically simple. This means that the number of associated functions of the operator L (if any exist) is finite.*

Now, let us proceed to finding the eigenfunctions of the operator L. The following theorem holds:

Theorem 3. *Under the conditions of Theorem 2, the eigenfunctions of the operator L satisfy the following asymptotic formula:*

$$y_k(x) = \cos\pi kx \ + O\left(\frac{1}{k}\right), \ k \to \infty. \tag{13}$$

Proof. Following [1, p. 84], we seek the eigenfunctions of the operator L in the form

$$y_k(x) = c_k \begin{vmatrix} y_1(x,\rho_k) & y_2(x,\rho_k) \\ U_2(y_1) & U_2(y_2) \end{vmatrix}, \tag{14}$$

if $\alpha_2 \neq \pm\beta_2$, or

$$y_k(x) = c_k \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ y_1(x,\rho_k) & y_2(x,\rho_k) \end{vmatrix}, \tag{15}$$

if $\alpha_1 \neq \pm\beta_1$, where c_k are some normalization factors to be determined. Assume $\alpha_2 \neq \pm\beta_2$. We have $\rho_k = \pi k + O\left(\frac{1}{k}\right)$ and

$$y_1(x,\rho_k) = \sin\rho_k x \ + O\left(\frac{1}{\rho_k}\right) = \sin\pi kx + O\left(\frac{1}{k}\right),$$

$$y_2(x,\rho_k) = \cos\rho_k x \ + O\left(\frac{1}{\rho_k}\right) = \cos\pi kx + O\left(\frac{1}{k}\right).$$

Substituting these expressions into determinant (14) and using formulas (11) and (12) for $\rho = \rho_k$, we obtain

$$y_k(x) = c_k \begin{vmatrix} \sin\pi kx + O\left(\frac{1}{k}\right) & \cos\pi kx + O\left(\frac{1}{k}\right) \\ \frac{1}{\rho_k}\left(-\beta_2\cos\rho_k \ + \alpha_2\right) + O\left(\frac{1}{\rho_k^2}\right) & \frac{1}{\rho}\beta_2\sin\rho_k \ + O\left(\frac{1}{\rho_k^2}\right) \end{vmatrix} =$$

$$c_k \begin{vmatrix} \sin\pi kx + O\left(\frac{1}{k}\right) & \cos\pi kx + O\left(\frac{1}{k}\right) \\ \frac{1}{\pi k}\left(-\beta_2(-1)^k + \alpha_2\right) + O\left(\frac{1}{k^2}\right) & \frac{1}{\rho}\beta_2 \sin\rho_k \ + O\left(\frac{1}{k^2}\right) \end{vmatrix} =$$

$$= c_k\left(\frac{1}{\pi k}\left(\alpha_2 - (-1)^k\beta_2\right)\cos\pi kx \ + O\left(\frac{1}{k^2}\right)\right).$$

By choosing $c_k = \frac{\pi k}{\left(\alpha_2 - (-1)^k\beta_2\right)}$, from the last equation, we obtain the validity of (13).
If $\alpha_2 = \pm\beta_2$, then from condition (10), it follows that $\alpha_1 \neq \pm\beta_1$. Proceeding similarly to the previous case and using formulas (11), (12), and the asymptotics of ρ_k, from formula (15) we obtain the validity of (13), where we should choose $c_k = \frac{\pi k}{\alpha_1 - (-1)^k\beta_1}$.

The theorem is proved.

4. Basis Property of Eigenfunctions

The operator L constructed in Section 2 does not have a dense domain in $L_p(0,1)$, and therefore the system of eigenfunctions and associated functions of this operator cannot be complete, let alone form a basis in this space. To address this issue, we consider the operator L not in the entire space $L_p(0,1)$, but in its closed subspace:

$$X_p = \{f(x) \in L_p(0,1) : U_\nu(f) = 0, \ \nu = 1,2\}.$$

Since the functionals $U_\nu, \ \nu = 1,2,$ are bounded in $L_p(0,1)$, it is clear that codim $X_p = 2$.
We define the operator L in the space X_p as follows: $D(L) = \left\{y \in W_p^2(0,1) \cap X_p : \ l(y) \in X_p \ \right\}$ and for $y \in D(L) : \ Ly = l(y)$.
It is evident that the operator defined in this way acts in the space X_p and has a dense domain in this space.
The following theorem was proved in [16].

Theorem 4. *[16] The eigenfunctions and associated functions of the operator L form a complete and minimal system in the space* $X_p, \ 1 \leq p < \infty$.

The main result of this work is the following theorem.

Theorem 5. *There exist functions* $\psi_k \in L_q(0,1), \ k = 1,2,$ *such that the system* $\{\psi_1, \psi_2\} \cup \{y_n\}_{n \geq 2}$ *forms a basis in the space* $L_p(0,1), \ 1 < p < \infty,$ *equivalent to the system* $\{\cos\pi nx\}_{n=0}^{\infty}$. *The system* $\{y_n\}_{n \geq 2}$ *of eigenfunctions and associated functions of the operator L forms a basis in the space* $X_p, \ 1 < p < \infty$.

Proof. Since codim $X_p = 2$, the space $L_p(0,1)$ can be represented as a direct sum: $L_p(0,1) = X_p \oplus X_0$, where dim $X_0 = 2$. According to the Hahn-Banach theorem, there exist functions $\psi_k \in L_q(0,1), \ k = 1,2,$ such that $\langle \varphi_i, \psi_k \rangle = \delta_{ik}, \ i,k = 1,2,$ and $\langle y, \psi_k \rangle = 0, \ \forall y \in X_p$, where $\varphi_1, \ \varphi_2-$are functions from the boundary conditions.

From Theorem 4, it follows that the system $\{\psi_1, \psi_2\} \cup \{y_n\}$ is complete and minimal in $L_p(0,1)$. Indeed, let $\{z_n\}$ be the system of eigenfunctions and associated functions of the adjoint operator L^*, forming a biorthogonal system with $\{y_n\}$. Then the system $\{\varphi_1, \varphi_2\} \cup \{z_n\}$ is biorthogonal to $\{\psi_1, \psi_2\} \cup \{y_n\}$, which is equivalent to the minimality of $\{\psi_1, \psi_2\} \cup \{y_n\}$ in $L_p(0,1)$. Now, we prove the completeness of this system. Suppose $z \in L_q(0,1)$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\langle \psi_1, z \rangle = \langle \psi_2, z \rangle = \langle y_n, z \rangle = 0$. For any $\forall f \in L_p(0,1)$ we represent it as $f = y + c_1\psi_1 + c_2\psi_2$, where $y \in X_p$. Then $\langle f, z \rangle = \langle y, z \rangle + c_1 \langle \psi_1, z \rangle + c_2 \langle \psi_2, z \rangle = \langle y, z \rangle$. On the other hand, since the system $\{y_n\}$ is complete in X_p, and $\langle y_n, z \rangle = 0$, $\forall n$, it follows that $\langle y, z \rangle = 0$, $\forall y \in X_p$. Thus, we obtain that $\langle f, z \rangle = 0$, $\forall f \in L_p(0,1)$, which implies $z = 0$, meaning that the system $\{\psi_1, \psi_2\} \cup \{y_n\}$ is complete in $L_p(0,1)$. Now we show that this system forms a basis in $L_p(0,1)$. From Theorem 3, it follows that the systems $\{y_n\}$ and $\{\cos\pi nx\}$ are $s-$close for any $s \in (1, \infty)$:

$$\sum_{n=0}^{\infty} \|y_n(x) - \cos\pi nx\|_{L_p}^s < +\infty, \tag{16}$$

where $y_0(x) = \psi_1(x), y_1(x) = \psi_2(x)$. Let $p \in (1, 2]$, then by the Hausdorff-Young inequality (see [21]), for all $\forall f \in L_p(0,1)$:

$$\left(\sum_{n=0}^{\infty} |\langle f, e_n \rangle|^q \right)^{\frac{1}{q}} \le C \|f\|_{L_p}, \tag{17}$$

where $e_n(x) = 2\cos\pi nx$. This inequality implies that the system $\{\cos\pi nx\}_{n=0}^{\infty}$ forms a $q-$basis in $L_p(0,1)$. For $p \in (2, \infty)$, we have $q \in (1, 2)$ and the continuous embedding $L_p(0,1) \subset L_q(0,1)$ holds. Again, using the Hausdorff-Young inequality: $\forall f \in L_p(0,1)$:

$$\left(\sum_{n=0}^{\infty} |\langle f, e_n \rangle|^p \right)^{\frac{1}{p}} \le C \|f\|_{L_q} \le C \|f\|_{L_q}, \tag{18}$$

that is, the system $\{\cos\pi nx\}_{n=0}^{\infty}$ forms $p-$basis in $L_p(0,1)$. Thus, denoting $r = \max\{p, q\}$ and choosing in inequality (16) $s = r(r-1)^{-1}$, and taking into account inequalities (17) and (18), we obtain that for any $p \in (1, \infty)$, all the conditions of Theorem 2.1 are satisfied, according to which the system $\{y_n(x)\}_{n=0}^{\infty}$ forms a basis in $L_p(0,1)$, equivalent to the system $\{\cos\pi nx\}_{n=0}^{\infty}$. Moreover, this result, as well as the asymptotics of the eigenfunctions, dictates that the numbering of the eigenfunctions and associated functions of the operator L should be performed as $\{y_n(x)\}_{n=2}^{\infty}$. Now, it is easy to establish the basis property of the system in the space X_p. Let $y(x)$ be an arbitrary function from X_p. Since in this case $\langle y, \varphi_1 \rangle = \langle y, \varphi_2 \rangle = 0$, its expansion in the basis $\{y_n(x)\}_{n=0}^{\infty}$ in $L_p(0,1)$ takes the form:

$$y = \sum_{n=2}^{\infty} \langle y, z_n \rangle y_n.$$

Furthermore, from the completeness and minimality of the system $\{y_n(x)\}_{n=2}^{\infty}$ in X_p we conclude that such an expansion is unique, i.e., the system $\{y_n(x)\}_{n=2}^{\infty}$ forms a basis in X_p.

Corollary 1. *The system $\{y_n(x)\}_{n=2}^{\infty}$ is an $r-$ basis in the space X_p, $1 < p < \infty$, where $r = \max\{p, q\}$.*

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