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Construction of a Basis in L_p From Root Functions of a Differential Operator With Non-strongly Regular Boundary Conditions

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Abstract. We study a spectral problem for an ordinary differential equation of the second order with non-strengthened regular boundary conditions on a finite interval [0,1]. Such problems arise when solving a non-local boundary value problem for partial differential equations by the Fourier method. They arise, for example, when solving non-stationary diffusion problems with boundary conditions of the Samarskii-Ionkin type, or when solving a stationary diffusion problem with opposite flows on a part of an interval. The boundary conditions of this problem are regular, but not strengthened regular in the sense of Birkhoff. The system of eigenfunctions of such a problem is complete and minimal, but does not form a basis in the space $L_p\,[0,1]$. In this case, direct application of the Fourier method is impossible. Based on these eigenfunctions, a new system of functions is constructed, which already forms a basis in $L_p\,[0,1]$.

Key Words and Phrases: non-strongly regular boundary conditions, eigenfunctions, almost normalized system, uniform minimality, basis.
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1. Introduction

The solution of some elliptic equations with nonlocal boundary conditions using the Fourier method leads to spectral problems with boundary conditions that are regular but not strongly regular. For this reason, the root functions of these problems do not generally form a basis in the corresponding function space. In such a case, direct application of the Fourier method is impossible. Based on these eigenfunctions, a new system of functions is constructed consisting of linear combinations of root functions, which already forms a basis in $L_p\,[0,1]$. However, the resulting system is not a system of eigenfunctions of the spectral problem. Nevertheless, this system is used to solve the equation under consideration by the Fourier method. One of such problems is the following initial-boundary value problem for the parabolic equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},\ 0 < x < 1,\ t > 0,$$

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with initial condition

$$U(x,0) = \varphi(x),\ 0 \le x \le 1,$$

and boundary conditions

$$U(0,t) = 0,\ \frac{\partial U}{\partial x}(0,t) = \frac{\partial U}{\partial x}(1,t) + \alpha U(1,t),\ t \ge 0.$$

This problem leads to the following spectral problem

$$\left.\begin{array}{c} -u''(x) = \lambda\, u(x),\ 0 < x < 1, \\ u(0) = 0,\quad u'(0) - u'(1) + \alpha\, u(1) = 0. \end{array}\right\} \tag{1}$$

The boundary conditions of this spectral problem are regular, but not strongly regular. A number of works by the authors [1-5] are devoted to the study of such problems in the Lebesgue space $L_2[0,1]$. It should be noted that issues related to this topic in the case of $\alpha = 0$ were also considered in works [6-11]. All these spectral problems are not self-adjoint. The case of $\alpha = 0$ differs from the case of $\alpha \neq 0$ in that in the first case all eigenvalues are double and they correspond to one eigenfunction and one associated function, and together they form a basis in $L_2[0,1]$. In the second case, all eigenvalues are simple, but the corresponding eigenfunctions are not a basis in $L_2[0,1]$. One of the methods for constructing a basis, based on the system of eigenfunctions of problem (1) in the case of $\alpha > 0$ was proposed in [1]. Using the eigenfunctions of this problem, a special system of functions is constructed, which will form a basis in $L_2[0,1]$. And this fact is applied to solve a nonlocal initial-boundary value problem for the heat equation. It is used in [3] to solve an inverse nonlocal boundary value problem for the heat equation, and in [4] to solve a nonlocal boundary value problem for the Helmholtz operator in a semicircle. A similar method was used in [5] to study the classical solvability of one nonlocal boundary value problem for the Laplace equation in a semicircle.

The aim of this work is to construct a basis in $L_p[0,1]$ from the system of eigenfunctions of problem (1) for any complex value of the parameter α.

2. Preliminaries

Let us present briefly the main definitions and facts which will be used in what follows. Let X be a Banach space. A system$\{x_n\}_{n\in N}$ of elements X is said to be complete in X if $\overline{L(\{x_n\}_{n\in N})} = X$; that is, any element of the space X can be approximated by a linear combination of elements of this system with any accuracy in the norm of the space X.

A system $\{x_n\}_{n\in N}$ of elements X is said to be minimal in X if $x_n \notin \overline{L\left(\{x_k\}_{k\neq n}\right)}$. It is well known that a system $\{x_n\}_{n\in N}$ is minimal if and only if there exists a biorthogonal system which is dual to it, that is, a system of linear functionals $\{x_n^*\}_{n\in N}$ from X^* such that $\langle x_n, x_k^*\rangle = \delta_{nk}$ for all $n,\ k \in N$. Moreover, if the initial system is complete and minimal in X, then the biorthogonal system is uniquely defined.

We say that a system $\{x_n\}_{n\in N}$ is uniformly minimal in X, if there exists $\gamma > 0$ such that for all $n \in N$,

$$\operatorname{dist}(x_n, X_n) \geq \gamma \|x_n\|_X \ ,$$

where $X_n = \overline{L\left[\{x_k\}_{k\neq n}\right]}$. It is also well known that a complete and minimal system $\{x_n\}_{n\in N}$ is uniformly minimal in X if and only if:

$$\sup_{n\in N} \|x_n\|_X \|x_n^*\|_{X^*} < \infty \ .$$

A system $\{x_n\}_{n\in N}$ forms a basis of the space X if, for any element $x \in X$, there exists a unique expansion into a series

$$x = \sum_{n=1}^{\infty} c_n x_n$$

converging in the norm of the space X.

Two systems $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ of a Banach space X are called equivalent if there exists an automorphism $T\ : X \to X$ that maps one of these systems to the other: $Tx_n = y_n$, $\forall n \in N$. A system equivalent to a basis is itself a basis in the same space.

A system in a Hilbert space that is equivalent to an orthonormal basis is called a Riesz basis. A Riesz basis is also an unconditional basis, i.e. it remains a basis under any permutation of its elements.

A system $\{x_n\}_{n\in N}$ is called a basis with brackets in a Banach space X if there exists a sequence $\{n_k\}_{k\in N}$ of positive integers such that $n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$, and for any $x \in X$ there is a unique expansion into a series

$$x = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i x_i, \quad (n_0 = 0)$$

converging in the norm of the space X. In the case of a Hilbert space, an unconditional basis with brackets is also called a Riesz basis with brackets.

We say that a system $\{x_n\}_{n\in N}$ is almost normalized in X, if

$$0 < \inf_{n\in N} \|x_n\| \leq \sup_{n\in N} \|x_n\| < \infty \quad .$$

A uniformly minimal system is almost normalized if and only if its bioorthogonal system is almost normalized.

Statement 1. *Let $\{x_n\}_{n\in N}$ be a minimal system in a Banach space X, $\{x_n^*\}_{n\in N}$ be its biorthogonal system. If the system $\{x_n\}_{n\in N}$ has two asymptotically close subsystems, i.e. there exist subsystems $\{x_{n_k}\}_{k\in N}$ and $\left\{x_{n'_k}\right\}_{k\in N}$ such that*

$$\lim_{k\to\infty} \left\|x_{n_k} - x_{n'_k}\right\|_X = 0, \tag{2}$$

then the system $\{x_n^*\}_{n\in N}$ is not almost normalized.

Proof. By $\{x^*_{n_k}\}_{k\in N}$ and $\left\{x^*_{n'_k}\right\}_{k\in N}$ we denote the corresponding subsystems of the biorthogonal system $\{x^*_n\}_{n\in N}$. Then from the biorthonormality conditions we have $\langle\langle x_{n_k}, x^*_{n_k}\rangle = 1,\ \left\langle x_{n'_k}, x^*_{n_k}\right\rangle = 0.$. From here we get $\langle\left\langle x_{n_k} - x_{n'_k}, x^*_{n_k}\right\rangle = 1$. Then

$$1 = \left|\left\langle x_{n_k} - x_{n'_k}, x^*_{n_k}\right\rangle\right| \le \left\|x_{n_k} - x_{n'_k}\right\|_X \left\|x^*_{n_k}\right\|_{X^*}$$

or

$$\left\|x^*_{n_k}\right\|_{X^*} \ge \left(\left\|x_{n_k} - x_{n'_k}\right\|_X\right)^{-1}.$$

Then from condition (2) it follows that

$$\lim_{k\to\infty} \left\|x^*_{n_k}\right\|_{X^*} = \infty. \tag{3}$$

Similarly, it is established that $\lim_{k\to\infty}\left\|x^*_{n'_k}\right\|_{X^*} = \infty$. Consequently, the system $\{x^*_n\}_{n\in N}$ is not almost normalized.

Statement 2. *If the system $\{x_n\}_{n\in N} \subset X$ is almost normalized and has two asymptotically close subsystems, then it is not uniformly minimal and, moreover, cannot be a basis in X.*

Proof. Let $\{x_{n_k}\}_{k\in N}$ and $\left\{x_{n'_k}\right\}_{k\in N}$ be asymptotically close subsystems of $\{x_n\}_{n\in N}$, and $\{x^*_{n_k}\}_{k\in N}$ and $\left\{x^*_{n'_k}\right\}_{k\in N}$ be the corresponding subsystems of the biorthogonal system $\{x^*_n\}_{n\in N}$. Then, from the condition of almost normalization of the system $\{x_n\}_{n\in N}$, we have: $\exists m > 0 :\ \|x_{n_k}\|_X > m,\ \forall k \in N$. Taking into account (3), we obtain

$$\lim_{k\to\infty} \|x_{n_k}\|_X \left\|x^*_{n_k}\right\|_{X^*} = \infty.$$

The latter means that the system $\{x_n\}_{n\in N}$ is not uniformly minimal.

Any basis is a complete and minimal system in X, and, therefore, we can uniquely find its biorthogonal dual system $\{x^*_n\}_{n\in N}$ and hence the expansion of any element $x \in X$ with respect to the basis $\{x\}_{n\in N}$ coincides with its biorthogonal expansion, that is, $c_n = \langle x, x^*_n\rangle$ for all $n \in N$.

We will use also some facts about p-closure bases. Concerning these facts more details one can see the works [12, 13].

Systems $\{x_n\}_{n\in N},\ \{y_n\}_{n\in N} \subset X$ in Banach space X are called p-closure if

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^p_X < \infty.$$

The minimal system $\{x_n\}_{n\in N} \subset X$ with biorthogonal system $\{x^*_n\}_{n\in N} \subset X^*$ is called p- besselian, if for any $x \in X$

$$\left(\sum_{n=1}^{\infty} |\langle x, x^*_n\rangle|^p\right)^{\frac{1}{p}} \le M\|x\|_X.$$

If the basis $\{x_n\}_{n\in N}$ for X is p- besselian, then we call it as p-basis.

It is valid the following

Theorem 1. *[12, 13] Let the system $\{x_n\}_{n\in N}$ is p -basis for Banach space X and the system $\{y_n\}_{n\in N} \subset X$ is p'– clouser to it, $1 < p < \infty$. Then the following assertions are equivalent:*

1. *$\{y_n\}_{n\in N}$ is complete in X;*
2. *$\{y_n\}_{n\in N}$ is minimal in X;*
3. *$\{y_n\}_{n\in N}$ is isomorphic to $\{x_n\}_{n\in N}$ basis for X.*

It is valid the following

Statement 3. *[14, 15] Let system $\{x_n\}_{n\in N}$ forms a basis with parentheses for Banach space X. If the system $\{x_n\}_{n\in N}$ is uniformly minimal and condition*

$$\sup_{k\in N} (n_{k+1} - n_k) < \infty \tag{4}$$

hold, then the system $\{x_n\}_{n\in N}$ forms a basis for X.

Statement 4. *[15] Let system $\{x_n\}_{n\in N}$ forms a Riesz basis with parentheses for Hilbert space X. If the system $\{x_n\}_{n\in N}$ is almost normalized, uniformly minimal and condition* (4) *hold, then it forms a basis Riesz for X.*

3. Study of the Spectral Problem

In this section we will study the properties of the eigenvalues and eigenfunctions of the following spectral problem

$$-u''(x) = \lambda u(x), \ 0 < x < 1, \ u(0) = 0, \ u'(0) = u'(1) + \alpha\, u(1), \tag{5}$$

where the parameter α can take any complex value. In the case $\alpha \neq 0$, the eigenvalues of the spectral problem can be divided into two series, which have the form

$$\lambda_{2k-1} = (\rho_{2k-1})^2, \ k \in N, \ \ \lambda_{2k} = (\rho_{2k})^2, \ k \in Z^+, \tag{6}$$

where $Z^+ = \{0\} \cup N$, $\rho_{2k-1} = 2\pi k$, and ρ_{2k} are the roots of the equation

$$tg\,\frac{\rho}{2} = \frac{\alpha}{\rho}. \tag{7}$$

Using the standard method we obtain that (see [16]) the following is true

Lemma 1. *Equation* (7) *for any complex α has a countable number of solutions that are asymptotically simple and have the asymptotics*

$$\rho_{2k} = 2\pi k + \frac{\alpha}{2\pi k} + O\left(\frac{1}{k^3}\right). \tag{8}$$

Each eigenvalue of problem (5) corresponds to a unique eigenfunction up to a non-zero factor. Using the numbering introduced by equalities (6), the set of eigenfunctions can be represented as

$$u_{2k-1}(x) = \sin 2\pi kx, \quad k \in N; \;\; u_{2k}(x) = \sin\rho_{k2}x, \quad k \in Z^+; \tag{9}$$

The problem conjugate to (5) is defined by the equality

$$-\vartheta''(x) = \lambda\vartheta(x), \; 0 < x < 1, \vartheta(0) = \vartheta(1), \vartheta'(1) + \alpha\,\vartheta(1) \; = \; 0. \tag{10}$$

It has the same eigenvalues (6) as problem (5). The corresponding eigenfunctions have the form

$$\vartheta_{2k-1}(x) = C_{2k-1}\left(\cos 2\pi kx \; - \frac{\alpha}{2\pi k}\sin 2\pi kx\right), \; k \in N; \tag{11}$$

$$\vartheta_{2k}(x) = C_{2k}\left(\cos\rho_{2k}x \; + \frac{\alpha}{\rho_{2k}}\sin\rho_{2k}x\right), \; k \in Z^+,$$

where

$$C_{2k-1} = -\frac{4\pi k}{\alpha}\;, \quad C_{2k} = \frac{4\pi k}{\alpha} + O\left(\frac{1}{k}\right).$$

The systems of eigenfunctions of problems (5) and (10) are numbered in such a way that $\langle u_n, \vartheta_m\rangle = \delta_{nm}$. The constants C_n are chosen so that $\langle u_n, \vartheta_n\rangle = 1, \;\; n \in Z^+$.

Let's show that the system $\{u_n(x)\}_{n\in Z^+}$ is not uniformly minimal in $L_p(0,1)$.

Theorem 2. *The system of eigenfunctions $\{u_n(x)\}_{n\in Z^+}$ of problem (5) is complete, minimal and almost normalized, but is not uniformly minimal in $L_p(0,1), 1 < p < \infty$.*

Proof. The spectral problem (5) is regular, but not strongly regular in the sense of Birkhoff (see [16]). From the results of [17], in particular, it follows that the eigenfunctions and associated functions of problem (5) form a basis with brackets in $L_p(0,1), 1 < p < \infty$. From this, in particular, follows the completeness of the system $\{u_n(x)\}_{n\in Z^+}$ in the space $L_p(0,1), \; 1 < p < \infty$. The system $\{\vartheta_n(x)\}_{n\in N}$ is a biorthogonal to $\{u_n(x)\}_{n\in Z^+}$ system regarding the space $L_p(0,1), \; 1 < p < \infty$, and therefore the system $\{u_n(x)\}_{n\in Z^+}$ is minimal in $L_p(0,1)$.

Let us show the almost normalized nature of the system $\{u_n(x)\}_{n\in Z^+}$. Let $1 < p < \infty$. We denote $2\delta_k = \rho_{2k} - 2\pi k$. Then from (8) we have $2\delta_k = \frac{\alpha}{2\pi k} + O\left(\frac{1}{k^3}\right)$ or $\delta_k = O\left(\frac{1}{k}\right)$. From here for the eigenfunctions $u_{2k}(x)$ we obtain

$$\sin\rho_{2k}x \; = \sin(2\pi k + 2\; delta_k)\,x \; = \sin 2\pi kx \; + O\left(\frac{1}{k}\right). \tag{12}$$

Let's estimate the norms of eigenfunctions:

$$\|u_{2k-1}\|_{L_p} = \left(\int_0^1 |\sin 2\pi kx\;|^p dx\right)^{\frac{1}{p}} \leq 1;$$

$$\|u_{2k}\|_{L_p} = \left(\int_0^1 |\sin\left(2\pi k + 2\delta_k\right)x\ |^p dx\right)^{\frac{1}{p}} \leq 1 + O\left(\frac{1}{k}\right).$$

From this we get

$$\overline{\lim_{k\to\infty}} \|u_{2k}\|_{L_p} \leq 1\ . \tag{13}$$

For the lower bound, we first consider the case $1 < p \leq 2$. Then for $u_{2k-1}\left(x\right)$ we have

$$\|u_{2k-1}\|_{L_p}^p = \int_0^1 |\sin 2\pi kx\ |^p dx \geq \int_0^1 \sin^2 2\pi kx\ dx = \frac{1}{2}.$$

It follows from this

$$\|u_{2k-1}\|_{L_p} \geq \left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

Similarly, for large values of k for the functions $u_{2k-1}\left(x\right)$ we obtain

$$\|u_{2k}\|_{L_p} = \left(\int_0^1 |\sin\left(2\pi k + 2\delta_k\right)\ x|^p dx\right)^{\frac{1}{p}} \geq$$

$$\geq \left(\int_0^1 |sin 2\pi kx\ |^p dx\right)^{\frac{1}{p}} - O\left(\frac{1}{k}\right)$$

$$\geq \left(\frac{1}{2}\right)^{\frac{1}{p}} - O\left(\frac{1}{k}\right) \to \left(\frac{1}{2}\right)^{\frac{1}{p}}, k \to \infty.$$

Hence,

$$\underline{\lim_{k\to\infty}} \|u_{2k}\|_{L_p} \geq \left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

From here, taking into account (13), we obtain the almost normalized nature of the system $\{u_n\left(x\right)\}_{n\in Z^+}$ for $1 < p \leq 2$.

Now let $p > 2$. Then we have a continuous embedding $L_p\left(0,1\right) \subset L_2\left(0,1\right)$ and

$$\|u_{2k-1}\|_{L_p} \geq \|u_{2k-1}\|_{L_2} = \left(\frac{1}{2}\right)^{\frac{1}{2}};$$

and also for large values of k

$$\|u_{2k}\|_{L_p} \geq \|u_{2k}\|_{L_2} \geq \|u_{2k-1}\|_{L_2} - O\left(\frac{1}{k}\right) \geq \left(\frac{1}{2}\right)^{\frac{1}{2}} - O\left(\frac{1}{k}\right).$$

From this we have

$$\underline{\lim_{n\to\infty}} \|u_{2k}\|_{L_p} \ \geq \left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

Thus, for all $p \in (1, \infty)$

$$0<\inf_{n\in N}\|u_n\|_{L_p} \le \sup_{n\in N}\|u_n\|_{L_p} < \infty,$$

i.e. the system $\{\ \{u_n(x)\}_{n\in Z^+}$ is almost normalized in $L_p(0,1)$.

Let us now proceed to the proof of the last statement of the lemma. From the asymptotics (12) we have

$$u_{2k}(x)-u_{2k-1}(x)=O\left(\frac{1}{k}\right).$$

Hence$\|u_{2k}-u_{2k-1}\|_{L_p}=O\left(\frac{1}{k}\right)$, i.e. the subsystems $\{u_{2k-1}(x)\}_{k\in N}$ and $\{u_{2k}(x)\}_{k\in N}$ are asymptotically close. Then it follows from Statement 1 that

$$\lim_{k\to\infty}\|\vartheta_{2k-1}\|_{L_{p'}} = \lim_{k\to\infty}\|\vartheta_{2k}\|_{L_{p'}}=\infty. \quad (14)$$

On the other hand, the system $\{u_n(x)\}_{n\in Z^+}$ is almost normalized in $L_p(0,1)$, so from Statement 1.2 we obtain that the system $\{u_n(x)\}_{n\in Z^+}$ is not uniformly minimal. Note that the validity of relations (14) can also be obtained directly from the explicit formulas (11) for the functions $\vartheta_n(x)$. The lemma is proven.

From this lemma follows

Corollary 1. *The system $\{u_n(x)\}_{n\in Z^+}$ does not form a basis for $L_p(0,1)$, $1<p<\infty$.*

4. Main results

Let us consider the case $\alpha=0$ separately. In this case, the spectral problem will take the form

$$-w''(x)=\lambda w(x),\ 0<x<1,\ w(0)=w'(0)-w'(1)=0. \quad (15)$$

In obtaining the main results we essentially will use the basicity in $L_p\ (0,\pi)$ the system $\{w_n(x)\}_{n\in Z^+}$ where

$$w_0(x)=x,\quad w_{2k-1}(x)=\sin 2\pi kx,\quad w_{2k}(x)=x\cos 2\pi kx\ ,\ k\in N,$$

which is a collection of root functions of the spectral problem (15).

It is valid

Theorem 3. *The system $\{w_n(x)\}_{n\in N}$ forms a $q-$basis for $L_p\ (0,1)$, $1<p<+\infty$, where $q=\max\{p,\ p'\}$. In the case $p=2$ this system is a Riesz basis for $L_2\ (0,1)$.*

Proof. As in the case of spectral problem (5), spectral problem (15) is also not strongly regular, and from the results of [17] it follows that the system $\{w_n(x)\}_{n\in Z^+}$of eigen and associated functions of this problem forms a basis with brackets in $L_p\ (0,1)$, $1<p<\infty$, and in brackets you need to combine pairs of terms corresponding to w_{2k-1} and w_{2k}, that is $n_{k+1}-n_k=2$.

The problem conjugate to (15) has the form

$$-z''(x)=\lambda\ z(x),\quad 0<x<1,\quad z'(1)=z(0)-z(1)=0. \quad (16)$$

The system of eigen and associated functions of the spectral problem (16) is the system$\{z_n(x)\}_{n\in Z^+}$, where

$$z_0(x)=2,\ z_{2k-1}(x)=4(1-x)\sin 2\pi kx,\quad z_{2k}(x)=4\cos 2\pi kx,\quad k\in N.$$

The systems $\{w_n(x)\}_{n\in Z^+}$ and $\{z_n(x)\}_{n\in Z^+}$ are biorthonormal, i.e.

$$\langle w_n,z_m\rangle=\delta_{nm},\quad \forall n,m\in Z^+.$$

From the formulas for $w_n(x)$ and $z_n(x)$ it is obvious that

$$\sup_{n\in Z^+}\|w_n\|_{L_p(0,1)}\ \|z_n\|_{L_{p'}(0,1)}<+\infty.$$

Thus, all the conditions of Statement 3 are satisfied, according to which the system $\{w_n(x)\}_{n\in Z^+}$ forms a basis in the space $L_p(0,1)$, $1<p<\infty$. Let us show that the system $\{w_n(x)\}_{n\in Z^+}$ is also a $q-$basis in this space, where $q=\max\{p,\ p'\}$. Let $p\in(1,2]$, then $q=p'$ and, as follows from the Hausdorff-Young inequality (see [18]), for any function $f(x)$ from $L_p(0,1)$ we have

$$\left(\sum_{k=0}^{\infty}|\langle f,z_{2k}\rangle|^{p'}\right)^{\frac{1}{p'}}\le C\|f\|_{L_p};$$

$$\left(\sum_{k=1}^{\infty}|\langle f,z_{2k-1}\rangle|^{p'}\right)^{\frac{1}{p'}}=\left(\sum_{k=1}^{\infty}\left|\int_0^1 f(x)\,4(1-x)\sin 2\pi kx\ \ dx\right|^{p'}\right)^{\frac{1}{p'}}\le$$

$$\le 4\left(\sum_{k=1}^{\infty}\left|\int_0^1\tilde f(x)\sin 2\pi kx\ \ dx\right|^{p'}\right)^{\frac{1}{p'}}\le 4C\left\|\tilde f\right\|_{L_p}\le 4C\|f\|_{L_p},$$

where$\tilde f(x)=(1-x)f(x)$ is denoted. Hence the system $\{w_n(x)\}_{n\in Z^+}$ is a $p'-$basis in $L_p(0,1)$.

If $p\in(2;+\infty)$, then $p'\in(1;2)$ and $q=p$, and again applying the Hausdorff-Young inequality and taking into account the embedding $L_p(0,1)\subset L_{p'}(0,1)$, we obtain

$$\left(\sum_{n=0}^{\infty}|\langle f,z_n\rangle|^{p}\right)^{\frac{1}{p}}\le C\|f\|_{L_{p'}}\le C\|f\|_{L_p},$$

i.e. the system $\{w_n(x)\}_{n\in Z^+}$ is a p -basis in $L_p(0,1)$**.**

Consider the case $p=2$. According to the results of [19], the system $\{\ \{w_n(x)\}_{n\in Z^+}$ forms a Riesz basis with brackets in $L_2(0,1)$, where the lengths of the brackets are uniformly bounded ($n_{k+1}-n_k=2,\ \forall k\in N$). In addition, it follows from the previous reasoning that this system is almost normalized and uniformly minimal in $L_2(0,1)$. Thus, all the conditions of Statement 1.4 are satisfied, according to which the system $\{w_n(x)\}_{n\in Z^+}$ forms a Riesz basis in the space $L_2(0,1)$. Theorem is proved.

Let us return to the case $\alpha \neq 0$. As shown above, in this case the eigenfunctions of the spectral problem (5) do not form a basis in any space $L_p(0,1)$, $1<p<\infty$. However, from the linear combinations of the elements of this system, it is possible to compose a new system, which will already be a basis in $L_p(0,1)$, and, accordingly, a Riesz basis in $L_2(0,1)$.

Following the work [1] we introduce to the consideration the following system

$$\varphi_{2k-1}(x) = u_{2k-1}(x)\,;\varphi_{2k}(x) = (u_{2k}(x) - u_{2k-1}(x))\,(2\delta_k)^{-1}, \forall k \in N, \tag{17}$$

which is a linear combination of the system $\{u_n(x)\}_{n\in N}$. It is valid the following

Theorem 4. *The system $\{\varphi_n\}_{n\in Z^+}$ forms an equivalent to the system $\{w_n\}_{n\in Z^+}$ basis for $L_p(0,1)$, $1<p<\infty$, with biorthogonal system $\{\psi_n\}_{n\in Z^+}$ where*

$$\psi_{2k-1} = \vartheta_{2k} + \vartheta_{2k-1}, \psi_{2k} = 2\delta_k\vartheta_{2k}, \forall k \in N. \tag{18}$$

In particular, for $p=2$ the system $\{\varphi_n\}_{n\in Z^+}$ forms a Riesz basis in $L_2(0,1)$.

Proof. Let us show that the system of functions $\{\varphi_n\}_{n\in Z^+}$ forms a basis in $L_p(0,1)$, $1<p<\infty$. It is obvious that it is complete and minimal in this space. Completeness follows from the completeness of the system $\{u_n\}_{n\in Z^+}$ in $L_p(0,1)$. The minimality of this system follows from the fact that it has a biorthogonal system $\{\psi_n\}_{n\in Z^+}$, defined by formula (18), which is verified directly.

From formulas (17) we have

$$\varphi_{2k-1}(x) - w_{2k-1}(x) = 0;$$

$$\varphi_{2k}(x) = \frac{1}{2\delta_k}\left(\sin\left((2\pi k + 2\delta_k)x\right) - \sin 2\pi kx\right) =$$

$$= \frac{\sin\delta_k x}{\delta_k x}\cdot x\cos\left((2\pi k+\delta_k)x\right) = (1+O(\delta_k))\,x\cos 2\pi kx\,\left(1+O\left(\delta_k^2\right)\right) =$$

$$= x\cos 2\pi kx + O(\delta_k) = w_{2k}(x) + O\left(\frac{1}{k}\right),$$

or

$$\varphi_{2k}(x) - w_{2k}(x) = O\left(\frac{1}{k}\right).$$

As a result, we obtain that for any $s,\ p\in(1,+\infty)$ we have

$$\sum_{n=0}^{\infty}\|\varphi_n - w_n\|_{L_p}^s < +\infty, \tag{19}$$

i.e. the systems $\{\varphi_n\}_{n\in Z^+}$ and $\{w_n\}_{n\in Z^+}$ are s-close in the space $L_p(0,1)$.

On the other hand, according to Theorem 3.1, the system $\{w_n\}_{n\in Z^+}$ is a $q-$ basis in $L_p(0,1)$, where $q=\max\{p,\ p'\}$. Choosing $s=q'$ in (19), we obtain that the systems $\{\varphi_n\}_{n\in Z^+}$ and $\{w_n\}_{n\in Z^+}$ are $q'-$close. Thus, all the conditions of Theorem 1.1 are satisfied and therefore the system $\{\varphi_n\}_{n\in Z^+}$ forms a basis in $L_p(0,1)$, equivalent to the basis $\{w_n\}_{n\in Z^+}$.

The second part of the theorem, which concerns the case $p = 2$, follows from the fact that according to Theorem 3.1 in this case the system $\{w_n\}_{n\in Z^+}$ is a Riesz basis in $L_2(0,1)$, and the system equivalent to the Riesz basis is itself a Riesz basis. The theorem is proved.

Corollary 2. *The system* $\{\varphi_n\}_{n\in Z^+}$ *is a* $q-$*basis in* $L_p(0,1)$, $1 < p < \infty$, *where* $q = max\{p,\ p'\}$.

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