Caspian Journal of Applied Mathematics, Ecology and Economics
V. 12, No 2, 2024, December
ISSN 1560-4055
https://doi.org/10.69624/1560-4055.12.2.30

Asymptotics of the Eigenvalues and Eigenfunctions of a Differential Operator with a conjugation conditions and a Summable Potential

A.Q.Ahmadov

Abstract. In this paper is studied the spectral problem for a discontinuous second order differential operator with a summabl potential function and a spectral parameter in conjugation conditions, that arises by solving the problem on vibrations of a loaded string with free ends. In the case of a summable potential function, asymptotic formulas for the eigenvalues and eigenfunctions of the spectral problem are obtained.

Key Words and Phrases: Eigenvalue, eigenfunction, asymptotic formulas.

2010 Mathematics Subject Classifications: Primary

Consider following spectral problem:

$$l\left(y\right)=-y^{"}\left(x\right)+q\left(x\right)y=\lambda y,\ x\in\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},1\right), \tag{1}$$

$$\left.\begin{array}{c} y^{'}(0)=y^{'}(1)=0,\\ y(\frac{1}{3}-0)=y(\frac{1}{3}+0),\\ y^{'}(\frac{1}{3}-0)-y^{'}(\frac{1}{3}+0)=m\lambda y\left(\frac{1}{3}\right), \end{array}\right\} \tag{2}$$

here, λ is spectral parameter, $q\left(x\right)$ is a complex-valued function summing over the interval $(0,1)$, m is complex nuber, and $m\neq 0$. Such spectral problems arise when the problem of vibrations of a loaded string with fixed ends is solved by applying the Fourier method [1-3]. The case of boundary conditions corresponding to a string with fixed ends (i.e. when instead of the boundary conditions $y^{'}(0)=y^{'}(1)=0$ in (2) $y(0)=y(1)=0$ are taken), is investigated in [4-10]. In [11], the asymptotic expressions for the eigenvalues and eigenfunctions of problem (1)–(2) in the case $q\left(x\right)$ were obtained, a linearization operator was constructed, and theorems on completeness and minimality were proved. Furthermore, [12,13] in the case $q\left(x\right)=0$ investigated the basis properties of the eigenfunctions of this problem in the spaces $L_p(0,1)\bigoplus C$ and Morrey spaces, respectively.

1. The asymptotic of eigenvalues

Let us denote $\lambda = \rho^2, \text{Im}\rho = \tau$. Also, let us denote by $y_1(x,\rho)$ the solution of equation (1) that satisfies the initial condition

$$\left.\begin{array}{l} y_1(0,\rho) = 1 \\ y_1'(0,\rho) = 0 \end{array}\right\} \tag{3}$$

in the segment $\left[0,\ \frac{1}{3}\right]$. Similarly let $y_2(x,\rho)$ be the solution that satisfies the initial condition

$$\left.\begin{array}{l} y_2(1,\rho) = 1 \\ y_2'(1,\rho) = 0 \end{array}\right\} \tag{4}$$

in the segment $\left[\ \frac{1}{3}, 1\right]$ of the same equation.

Lemma 1. The following formulas are true for the solutions $y_1(x,\rho)$ and $y_2(x,\rho)$ of the equation of (1) and their derivatives with respect to x.

$$y_1(x,\rho) = \cos\rho x + \frac{1}{\rho}\int_0^x q(t)\, y_1(t,\rho)\sin\rho(x-t)\,dt,\ 0 < x < \frac{1}{3}, \tag{5}$$

$$y_1'(x,\rho) = -\rho\sin\rho x + \int_0^x q(t)\, y_1(t,\rho)\cos\rho(x-t)\,dt,\ 0 < x < \frac{1}{3}, \tag{6}$$

$$y_2(x,\rho) = \cos\rho(1-x) - \frac{1}{\rho}\int_x^1 q(t)\, y_2(t,\rho)\sin\rho(x-t)\,dt, \qquad \frac{1}{3} < x < 1, \tag{7}$$

$$y_2'(x,\rho) = \rho\sin\rho(1-x) - \int_x^1 q(t)\, y_2(t,\rho)\cos\rho(x-t)\,dt, \qquad \frac{1}{3} < x < 1. \tag{8}$$

Proof. Since the function $y_1(x,\rho)$ is a solution of equation (1)

$$\int_0^x q(t)\, y_1(t,\rho)\ \sin\rho(x-t)\ dt =$$

$$= \int_0^x \sin\rho(x-t)\ y_1''(t,\rho)\,dt + \rho^2\int_0^x \sin\rho(x-t)\, y_1(t,\rho)\,dt. \tag{9}$$

If we intergarate the first integral on the right-hand side of the last equation twice by patrs and consider (3), we obtain following

$$\int_0^x q(t)\, y_1(t,\rho)\sin\rho(x-t)\,dt = \rho y_1(x,\rho) - \rho\cos\rho x. \tag{10}$$

That is, (5) is true. To get the equation (6), it is enough to differentiate the equation (5). Equations (7) and (8) are obtained by making similar calculations.

Lemma 2. When $\rho \to \infty$, the following asymptotic formulas hold true:

$$y_1(x,\rho) = \cos\rho\, x + O\left(\frac{e^{|\tau|x}}{|\rho|}\right),\quad x \in \left[0, \frac{1}{3}\right], \tag{11}$$

$$y_2(x,\rho) = cos\rho(1- \ x) + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|}\right), x \in \left[\frac{1}{3}, 1\right]. \tag{12}$$

Let us introduce the following functions to express the subsequent results:

$$q_1(x) = \frac{1}{2}\int_0^x q(t)\,dt, q_2(x) = \frac{1}{2}\int_x^1 q(t)\,dt \tag{13}$$

Theorem 1. The eigenvalues of problem (1)-(2) are asymptotically simple and consist of two series: $\lambda_{i,n} = \rho_{i,n}^2, i = 1,2;\ n \in Z^+,\ Z^+ = N \cup \{\emptyset\}$ and the following asymptotic expressions hold for $\rho_{i,n}$.

$$\left.\begin{array}{l} \rho_{1,n} = 3\pi n + \frac{3\pi}{2} + O\left(\frac{1}{n}\right), \\ \rho_{2,n} = \frac{3\pi n}{2} + \frac{3\pi}{4} + O\left(\frac{1}{n}\right). \end{array}\right\} \tag{14}$$

Proof. If we substitute the asymptotic expression of $y_1(x,\rho)$ from (11) into the right-hand side of (5), we obtain following:

$$y_1(x,\rho) = \cos\rho x + \frac{1}{\rho}\int_0^x q(t)\sin\rho(x-t)\left[\cos\rho t + O\left(\frac{e^{|\tau|}}{\rho}\right)\right]dt =$$

$$= \cos\rho x + \frac{1}{\rho}\int_0^x q(t)\sin\rho(x-t)\cos\rho t dt +$$

$$+\frac{1}{\rho^2}\int_0^x q(t)\sin\rho(x-t)\ O\left(e^{|\tau|t}\right)dt =$$

$$= \cos\rho x + \frac{1}{2\rho}\int_0^x q(t)\left[\sin\rho(x-2t) + \sin\rho x\right]dt +$$

$$+\frac{1}{\rho^2}\int_0^x q(t)\sin\rho(x-t)\ O\left(e^{|\tau|t}\right) =$$

$$= \cos\rho x + \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\,dt + \frac{\sin\rho x}{2\rho}\int_0^x q(t)\,dt +$$

$$+\frac{1}{\rho^2}\int_0^x q(t)\sin\rho(x-t)\ O\left(e^{|\tau|t}\right)dt = \cos\rho x + \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\,dt +$$

$$+\frac{1}{\rho}\sin\rho x\left(\frac{1}{2}\int_0^x q(t)\,dt\right) + \frac{e^{|\tau|x}}{\rho^2}\int_0^x \frac{\sin\rho(x-t)}{e^{|\tau|(x-t)}}dt.$$

Therefor,

$$y_1(x,\rho) = \cos\rho x + \frac{1}{\rho}q_1(x)\sin\rho x +$$

$$+\frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\,dt + O\left(\frac{e^{|\tau|x}}{|\rho|^2}\right) \tag{15}$$

is true.

Also, if we substitute the asymptotic expression of $y_1(x,\rho)$ from (11) into the right-hand side of the equation (6), we obtain:

$$y_1'(x,\rho) = -\rho\sin\rho x + \int_0^x q(t)\, y_1(t,\rho)\cos\rho(x-t)\,dt =$$

$$= -\rho\sin\rho x + \int_0^x q(t)\cos\rho(x-t)\left[\cos\rho t + O\left(\frac{e^{|\tau|t}}{|\rho|}\right)\right]dt =$$

$$= -\rho\sin\rho x + \frac{1}{2}\int_0^x q(t)\left[\cos\rho x + \cos\rho(2t-x)\right]dt +$$

$$+ \int_0^x q(t)\cos\rho(x-t)\; O\left(\frac{e^{|\tau|t}}{|\rho|}\right)dt = -\rho\sin\rho x +$$

$$+\frac{1}{2}\int_0^x q(t)\cos\rho x\,dt + \frac{1}{2}\int_0^x q(t)\cos\rho(x-2t)\,dt +$$

$$+ \int_0^x q(t)\cos\rho(x-t)\; O\left(\frac{e^{|\tau|t}}{|\rho|}\right)dt = -\rho\sin\rho x + q_1(x)\cos\rho x +$$

$$+\frac{1}{2}\int_0^x q(t)\cos\rho(x-2t)\,dt + O\left(\frac{e^{|\tau|x}}{|\rho|}\right)\int_0^x q(t)\,\frac{\cos\rho(x-t)}{e^{|\tau|(x-t)}}\;O(1)\,dt =$$

$$= -\rho\sin\rho x + q_1(x)\cos\rho x + \frac{1}{2}\int_0^x q(t)\cos\rho(x-2t)\,dt + O\left(\frac{e^{|\tau|x}}{|\rho|}\right).$$

Thus,

$$y_1'(x,\rho) = -\rho\sin\rho x + q_1(x)\cos\rho x +$$

$$+\frac{1}{2}\int_0^x \cos\rho(x-2t)\cdot q(t)\,dt + O\left(\frac{e^{|\tau|x}}{|\rho|}\right). \tag{16}$$

By similar calculations, we obtain the following asymptotic equalities for $y_2(x,\rho)$ and $y_2'(x,\rho)$:

$$y_2(x,\rho) = \cos\rho(1-x) + \frac{1}{\rho}\cdot q_2(x)\sin\rho(1-x) +$$

$$+\frac{1}{2\rho}\int_x^1 \sin\rho(2t-x-1)\cdot q(t)\,dt + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|^2}\right), \tag{17}$$

$$y_2'(x,\rho) = \rho\sin\rho(1-x) - q_2(x)\cdot\cos\rho(1-x) -$$

$$-\frac{1}{2}\int_x^1 \cos\rho(2t-x-1)\cdot q(t)\,dt + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|}\right). \tag{18}$$

The solution $y(x,\rho)$ of problem (1)-(2) should be in the following form:

$$y(x,\rho)=\begin{cases} C_1y_1(x,\rho), & 0<x<\frac{1}{3}, \\ C_2y_2(x,\rho), & \frac{1}{3}<x<1, \end{cases} \tag{19}$$

Here C_1 and C_2 are complex numbers. Therefore, the function satisfies the conditions given in (2). Now, let us impose the requirement that it also satisfies the conditions in (3) and (4). In this case, to determine the coefficients C_1 and C_2 we obtain the following system:

$$\begin{cases} C_1y_1\left(\frac{1}{3},\rho\right)-C_2y_2\left(\frac{1}{3},\rho\right)=0 \\ C_1{y'}_1\left(\frac{1}{3},\rho\right)-C_2{y'}_2\left(\frac{1}{3},\rho\right)=C_1\rho^2my_1\left(\frac{1}{3},\rho\right) \end{cases} \tag{20}$$

Taking into account the expressions (15), (16), (17), and (18) in (20), we obtain:

$$\begin{cases} C_1\left(\cos\frac{1}{3}\rho+\frac{1}{\rho}q_1\sin\frac{1}{3}\rho+\frac{1}{2\rho}\int_0^{1/3}\sin\rho\left(\frac{1}{3}-2t\right)q(t)\,dt+O\left(\frac{e^{1/3|\tau|}}{|\rho|^2}\right)\right)- \\ -C_2\left(\cos\frac{2}{3}\rho-\frac{1}{\rho}q_2\sin\frac{2}{3}\rho+\frac{1}{2\rho}\int_{1/3}^{1}\sin\rho\left(2t-\frac{4}{3}\right)q(t)\,dt+O\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right)\right)=0 \\ C_1\left(-\rho\sin\frac{1}{3}\rho+q_1\cos\frac{1}{3}\rho+\frac{1}{2}\int_0^{1/3}\cos\rho\left(2t-\frac{1}{3}\right)q(t)\,dt+O\left(\frac{e^{|\tau|/3}}{|\rho|}\right)\right)- \\ -C_2\left(\rho\sin\frac{2}{3}\rho-q_2\cos\frac{2}{3}\rho-\frac{1}{2}\int_{1/3}^{1}\cos\rho\left(2t-\frac{4}{3}\right)q(t)\,dt+O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right)\right)= \\ =C_1\rho^2m\left(\cos\frac{1}{3}\rho+\frac{1}{\rho}q_1\sin\frac{1}{3}\rho+\frac{1}{2\rho}\int_0^{1/3}\sin\rho\left(\frac{1}{3}-2t\right)q(t)\,dt+O\left(\frac{e^{1/3|\tau|}}{|\rho|^2}\right)\right). \end{cases}$$

Here,

$$q_1=q_1\left(\frac{1}{3}\right),q_2=q_2\left(\frac{1}{3}\right).$$

For the determination of the eigenvalues, we obtain the following equality:

$$\Delta\left(\rho^2\right)=\begin{vmatrix} a_{11}(\rho) & a_{12}(\rho) \\ a_{21}(\rho) & a_{22}(\rho) \end{vmatrix}=0,$$

Here

$$a_{11}\rho=\cos\frac{1}{3}\rho+\frac{1}{\rho}q_1\sin\frac{1}{3}\rho+\frac{1}{2\rho}\int_0^{1/3}\sin\rho\left(\frac{1}{3}-2t\right)q(t)\,dt+O\left(\frac{e^{1/3|\tau|}}{|\rho|^2}\right)$$

$$a_{12}(\rho)=-\cos\frac{2}{3}\rho+\frac{1}{\rho}q_2\sin\frac{2}{3}\rho-\frac{1}{2\rho}\int_{1/3}^{1}\sin\rho\left(2t-\frac{4}{3}\right)q(t)\,dt-O\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right)$$

$$a_{21}(\rho)=\left(-\rho\sin\frac{1}{3}\rho-\rho^2m\cos\frac{1}{3}\rho\right)+\left(q_1\cos\frac{1}{3}\rho-\rho mq_1\sin\frac{1}{3}\rho\right)+$$

$$\left(\frac{1}{2}\int_0^{1/3}\cos\rho\left(2t-\frac{1}{3}\right)q(t)\,dt-\frac{\rho m}{2}\int_0^{1/3}\sin\rho\left(\frac{1}{3}-2t\right)q(t)\,dt\right)+O\left(e^{1/3|\tau|}\right)$$

$$a_{22}(\rho)=-\rho\sin\frac{2}{3}\rho+q_2\cos\frac{2}{3}\rho+\frac{1}{2}\int_{1/3}^{1}\cos\rho\left(\frac{4}{3}-2t\right)q(t)\,dt-O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right).$$

For any arbitrary complex number z, by utilizing the inequalities

$$|\sin z|\leq e^{|\mathrm{Im} z|},\ |\cos z|\leq e^{|\mathrm{Im} z|},$$

the following results can be derived:

$$\begin{array}{ll}\left|\cos\rho\left(\frac{1}{3}-2t\right)\right|\leq e^{1/3|\tau|}, & 0\leq t\leq\frac{1}{3},\\ \left|\cos\rho\left(2t-\frac{4}{3}\right)\right|\leq e^{2/3|\tau|}, & \frac{1}{3}\leq t\leq 1,\\ \left|\sin\rho\left(2t-\frac{1}{3}\right)\right|\leq e^{1/3|\tau|}, & 0\leq t\leq\frac{1}{3},\\ \left|\sin\rho\left(\frac{4}{3}-2t\right)\right|\leq e^{2/3|\tau|}, & \frac{1}{3}\leq t\leq 1,\end{array}$$

Here $\mathrm{Im}\rho\ =\tau$ is denoted. As $|\rho|\to\infty$, by applying the previously mentioned inequalities, the following is obtained:

$$\int_0^{1/3}q(t)\cos\rho\ \left(\frac{1}{3}-2t\right)dt=O\left(e^{1/3|\tau|}\right),$$

$$\int_{1/3}^{1}q(t)\cos\rho\ \left(2t-\frac{4}{3}\right)dt=O\left(e^{2/3|\tau|}\right),$$

$$\int_0^{1/3}q(t)\sin\rho\ \left(2t-\frac{1}{3}\right)dt=O\left(e^{1/3|\tau|}\right),$$

$$\int_{1/3}^{1}q(t)\sin\rho\left(\frac{4}{3}-2t\right)\ dt=O\left(e^{2/3|\tau|}\right).$$

By applying the asymptotic formulas above, $\Delta\left(\rho^2\right)$ can be expressed as follows:

$$\Delta\left(\rho^2\right)=\begin{vmatrix}\cos\frac{1}{3}\rho & -\cos\frac{2}{3}\rho\\ -\rho\sin\frac{1}{3}\rho-\rho^2 m\cos\frac{1}{3}\rho & -\rho\sin\frac{2}{3}\rho\end{vmatrix}+$$

$$+\begin{vmatrix}\cos\frac{1}{3}\rho & \frac{1}{\rho}q_2\sin\frac{2}{3}\rho\\ -\rho\sin\frac{1}{3}\rho-\rho^2 m\cos\frac{1}{3}\rho & q_2\cos\frac{2}{3}\rho\end{vmatrix}+$$

$$+\begin{vmatrix}\cos\frac{1}{3}\rho & -\frac{1}{2\rho}\int_{\frac{1}{3}}^{1}\sin\rho\left(2t-\frac{4}{3}\right)\cdot q(t)\,dt\\ -\rho\sin\frac{1}{3}\rho-\rho^2 m\cos\frac{1}{3}\rho & \frac{1}{2}\int_{\frac{1}{3}}^{1}\cos\rho\left(\frac{4}{3}-2t\right)\cdot q(t)\,dt\end{vmatrix}+$$

$$+\begin{vmatrix} \cos\frac{1}{3}\rho & O\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right) \\ -\rho\sin\frac{1}{3}\rho-\rho^2 m\cos\frac{1}{3}\rho & O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right) \end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{\rho}q_1\sin\frac{1}{3}\rho & -\cos\frac{2}{3}\rho \\ q_1\cos\frac{1}{3}\rho-\rho m q_1\sin\frac{1}{3}\rho & -\rho\sin\frac{2}{3}\rho \end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{\rho}q_1\sin\frac{1}{3}\rho & \frac{1}{\rho}q_2\sin\frac{2}{3}\rho \\ q_1\cos\frac{1}{3}\rho-\rho m q_1\sin\frac{1}{3}\rho & q_2\cos\frac{2}{3}\rho \end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{\rho}q_1\sin\frac{1}{3}\rho & -\frac{1}{2\rho}\int_{\frac{1}{3}}^{1}\sin\rho\left(2t-\frac{4}{3}\right)\cdot q(t)\,dt \\ q_1\cos\frac{1}{3}-\rho m q_1\sin\frac{1}{3}\rho & \frac{1}{2}\int_{\frac{1}{3}}^{1}\cos\left(\frac{4}{3}-2t\right)\cdot q(t)\,dt \end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{\rho}q_1\sin\frac{1}{3}\rho & O\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right) \\ q_1\cos\frac{1}{3}\rho-\rho m q_1\sin\frac{1}{3}\rho & O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right) \end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{2\rho}\int_0^{\frac{1}{3}}\sin\rho\left(\frac{1}{3}-2t\right)q(t)\,dt & -\cos\frac{2}{3}\rho \\ \frac{1}{2}\int_0^{\frac{1}{3}}\cos\rho\left(2t-\frac{1}{3}\right)q(t)\,dt-\frac{\rho m}{2}\int_0^{1/3}\sin\rho\left(\frac{1}{3}-2t\right)q(t)\,dt & -\rho\sin\frac{2}{3}\rho \end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{2\rho}\int_0^{\frac{1}{3}}\sin\rho\left(\frac{1}{3}-2t\right)q(t)\,dt & \frac{1}{\rho}q_2\sin\frac{2}{3}\rho \\ \frac{1}{2}\int_0^{\frac{1}{3}}\cos\rho\left(2t-\frac{1}{3}\right)q(t)\,dt-\frac{\rho m}{2}\int_0^{\frac{1}{3}}\sin\rho\left(\frac{1}{3}-2t\right)q(t)\,dt & q_2\cos\frac{2}{3}\rho \end{vmatrix}+$$

$$+\begin{vmatrix} O\left(\frac{e^{1/3|\tau|}}{|\rho|^2}\right) & -\cos\frac{2}{3}\rho \\ O\left(e^{1/3|\tau|}\right) & \rho\sin\frac{2}{3}\rho \end{vmatrix}+O\left(\frac{e^{|\tau|}}{|\rho|}\right).$$

In the final expression, after expanding all the determinants and performing the corresponding calculations, the following expression for $\Delta\left(\rho^2\right)$ is obtained:

$$\Delta\left(\rho^2\right)=\cos^3\frac{1}{3}\rho\left(-2\rho^2 m+4q_1-2mq_1q_2\right)+$$

$$+\sin^3\frac{1}{3}\rho\left(4\rho-2\rho m q_2+2\rho m q_1\right)+$$

$$+\sin\frac{1}{3}\rho\left(-3\rho+2\rho m q_2-\rho m q_1-\frac{1}{\rho}q_1q_2\right)+$$

$$+\cos\frac{1}{3}\rho\left(\rho^2 m-3q_1+q_2+2mq_1q_2\right)+\sin\frac{1}{3}\rho\times$$

$$\times\left(\begin{array}{c}\frac{1}{2}\int_{\frac{1}{3}}^{1}q(t)\sin\rho\left(\frac{4}{3}-2t\right)dt-\frac{mq_1}{2}\int_{\frac{1}{3}}^{1}q(t)\sin\rho\left(2t-\frac{4}{3}\right)dt+O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right)+\\ -\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right)+\frac{1}{2\rho}q_1\int_{\frac{1}{3}}^{1}q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt+O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right)\end{array}\right)+$$

$$+\cos\frac{1}{3}\rho\left(\begin{array}{c}\frac{1}{2}\int_{\frac{1}{3}}^{1}q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt-O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right)+\frac{\rho m}{2}\int_{\frac{1}{3}}^{1}q(t)\sin\rho\left(\frac{4}{3}-2t\right)dt+\\ +\frac{q_1}{2\rho}\int_{\frac{1}{3}}^{1}q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt+O\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right)+O\left(e^{\frac{2}{3}|\tau|}\right)\end{array}\right)+$$

$$+\sin\frac{1}{3}\rho\cos\frac{1}{3}\rho\left(\begin{array}{c}\int_{0}^{\frac{1}{3}}q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt+q_2 m\int_{0}^{\frac{1}{3}}q(t)\sin\rho\left(\frac{1}{3}-2t\right)dt-\\ -\frac{1}{\rho}q_2\int_{0}^{\frac{1}{3}}q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt+O\left(\frac{e^{1/3|\tau|}}{|\rho|}\right)\end{array}\right)+$$

$$+\cos\frac{2}{3}\rho\left(\begin{array}{c}\frac{1}{2}\int_{0}^{\frac{1}{3}}q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt+\frac{1}{2\rho}q_2\int_{0}^{\frac{1}{3}}q(t)\sin\rho\left(\frac{1}{3}-2t\right)dt-\\ -\frac{\rho m}{2}\int_{0}^{\frac{1}{3}}q(t)\sin\rho\left(\frac{1}{3}-2t\right)dt+O\left(e^{\frac{1}{3}|\tau|}\right)\end{array}\right)+$$

$$+O\left(\frac{e^{|\tau|}}{|\rho|}\right) \tag{21}$$

Subsequently, we will consider that the parameter ρ varies within the strip $|Im\rho\,|\leq\alpha$. Under this condition, as $|\rho|\to+\infty$, the following asymptotic equalities hold:

$$\left.\begin{array}{c}O\left(\frac{e^{|\tau|}}{\rho}\right)=O\left(\frac{e^{1/3|\tau|}}{\rho}\right)=O\left(\frac{e^{2/3|\tau|}}{\rho}\right)=O\left(\frac{1}{\rho}\right)\\ O\left(\frac{e^{2/3|t|}}{\rho^2}\right)=O\left(\frac{1}{\rho^2}\right),\\ O\left(e^{|\tau|}\right)=O(1)\end{array}\right\} \tag{22}$$

On the other hand,as $|\rho|\to+\infty$ within the strip $|Im\rho\,|\leq\alpha$ the following relations hold:

$$\left.\begin{array}{c}\int_0^{1/3}q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt=o(1),\\ \int_{1/3}^{1}q(t)\cos\rho\ \ (2t-4/3)\,dt=o(1),\\ \int_0^{1/3}q(t)\sin\rho\ \ \left(2t-\frac{1}{3}\right)dt=o(1),\\ \int_{1/3}^{1}q(t)\sin\left(\frac{4}{3}-2t\right)\ dt=o(1),\end{array}\right\} \tag{23}$$

Taking (22) and (23) into account in (21), we obtain the following:

$$\Delta\left(\rho^2\right) = \rho\sin\frac{\rho}{3}\left(\beta_1+\beta_2\sin^2\frac{\rho}{3}+O\left(\frac{1}{\rho}\right)\right)+\rho^2 m\cos\frac{\rho}{3}-2\rho^2 m\cos^3\frac{\rho}{3}+O\left(\frac{1}{\rho}\right). \quad (24)$$

Here,

$$\beta_1=-3+2mq_1-mq_2, \beta_2=4+2mq_1-2mq_2$$

is denoted. From the resulting expression (23), based on Rouche's theorem, it is clear that the function $\Delta\left(\rho^2\right)$ has two series of zeros, $\rho_{1,n}$ and $\rho_{2,n}$ which are asymptotically close to the zeros of the functions $cos\frac{\rho}{3}$ and $cos\frac{2\rho}{3}$, respectively. Thus, the following asymptotic formulas hold for, $\rho_{1,n}$ and $\rho_{2,n}$:

$$\rho_{1,n}=3\pi n+\frac{3\pi}{2}+O\left(\frac{1}{n}\right),\ \rho_{2,n}=\frac{3\pi n}{2}+\frac{3\pi}{4}+O\left(\frac{1}{n}\right).$$

The estimate of the remainder term of the asymptotics in these formulas is obtained by the standard method (see [14]).

2. The asymptotic of eigenfunctions

We now proceed to determine the asymptotic formulas for the eigenfunctions associated with the problem (1)-(2).

Theorem 2. Suppose that the function $q(x)$ satisfies the conditions of Theorem 1. Then, for the eigenvalues $\lambda_{i,n}=(\rho_{1,n})^2, i=1,2; n\in N$, the corresponding eigenfunctions $y_{i,n}(x)$ satisfy the following asymptotic formulas:

$$y_{2,n}(x)=\begin{cases} cos\left(3\pi n+\frac{3\pi}{2}\right)x+O\left(\frac{1}{n}\right), & x\in\left[0,\frac{1}{3}\right], \\ O\left(\frac{1}{n}\right), & x\in\left[\frac{1}{3},1\right], \end{cases} \quad (25)$$

$$y_{2,n}(x)=\begin{cases} O\left(\frac{1}{n}\right), & x\in\left[0,\frac{1}{3}\right], \\ cos\left(\frac{3\pi n}{2}+\frac{3\pi}{4}\right)(1-x)+O\left(\frac{1}{n}\right), & x\in\left[\frac{1}{3},1\right], \end{cases} \quad (26)$$

Proof. First, let us determine the eigenfunction corresponding to the eigenvalue $\lambda_{1,n}$. To this end, let us substitute $\rho=\rho_{1,n}$ into equation (20) and choose:

$$C_{1,n}=-y_2\left(\frac{1}{3},\rho_{1,n}\right)$$

$$C_{1,n}=-\left(\cos\frac{2}{3}\rho_{1,n}+\frac{1}{\rho_{1,n}}q_2\sin\frac{2}{3}\rho_{1,n}\right)+O\left(\frac{1}{n}\right)=$$

$$=-cos\left(\pi+2\pi n+O\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n}\right)=1+O\left(\frac{1}{n}\right),$$

$$C_{2,n} = -y_1\left(\frac{1}{3},\rho_{1,n}\right) = -\left(\cos\frac{1}{3}\ \rho_{1,n} + \frac{1}{\rho_{1,n}}q_1\sin\frac{1}{3}\ \rho_{1,n}\right) + O\left(\frac{1}{n}\right) =$$

$$= -cos\left(\frac{\pi}{2} + \pi n + O\left(\frac{1}{n}\right)\right) + O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right).$$

Consequently, we obtain:

$$y_{1,n}(x) = \begin{cases} \left(1 + O\left(\frac{1}{n}\right)\right) y_1\left(x,\rho_{1,n}\right), & x \in \left[0,\frac{1}{3}\right] \\ O\left(\frac{1}{n}\right) y_2\left(x,\rho_{1,n}\right), & x \in \left[\frac{1}{3},1\right] \end{cases} =$$

$$= \begin{cases} cos\left(3\pi n + \frac{3\pi p}{2}\right) x + O\left(\frac{1}{n}\right), & x \in \left[0,\frac{1}{3}\right] \\ O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3},1\right] \end{cases}.$$

Now, let us determine the eigenfunction corresponding to the eigenvalue $\lambda_{2,n}$. To this end, we substitute $\rho = \rho_{2,n}$ into equation (20) and define

$$C_{1,n} = (-1)^n y_2\left(\frac{1}{3},\rho_{2,n}\right),\ C_{2,n} = (-1)^n y_1\left(\frac{1}{3},\rho_{2,n}\right).$$

Then, we obtain:

$$C_{1,n} = (-1)^n y_2\left(\frac{1}{3},\rho_{2,n}\right) = (-1)^n\left(\cos\frac{2}{3}\ \rho_{2,n} + \frac{1}{\rho_{1,n}}q_2\sin\frac{2}{3}\ \rho_{2,n}\right) + O\left(\frac{1}{n}\right) =$$

$$= (-1)^n cos\left(\pi n + \pi + O\left(\frac{1}{n}\right)\right) + O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right),$$

$$C_{2,n} = (-1)^n y_1\left(\frac{1}{3},\rho_{2,n}\right) = (-1)^n\left(\cos\frac{1}{3}\ \rho_{2,n} + \frac{1}{\rho_{1,n}}q_2\sin\frac{1}{3}\ \rho_{2,n}\right) + O\left(\frac{1}{n}\right) =$$

$$= (-1)^n cos\left(\pi n + \frac{\pi}{2} + O\left(\frac{1}{n}\right)\right) + O\left(\frac{1}{n}\right) = (-1)^n + O\left(\frac{1}{n}\right).$$

Consequently, we obtain:

$$y_{2,n}(x) = \begin{cases} O\left(\frac{1}{n}\right), & x \in \left[0,\frac{1}{3}\right], \\ cos\left(\frac{3pn}{2} + \frac{3p}{4}\right)(1-x) + O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3},1\right]. \end{cases}$$

The theorem is proven.

Acknowledgment. This work was supported by the Azerbaijan Science Foundation-Grant №AEF-MCG-2023-1(43)-13/06/1-M-06.

References

[1] A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* (Mosk. Gos. Univ., Moscow, 1999; Dover, New York, 2011).

[2] F.V. Atkinson, *Discrete and Continuous Boundary Problems.* Moscow, Mir, 1968.

[3] L. Collatz, Eigenvalue Problems. Moscow, Fizmatgiz, 1968, 504 p (in Russian)

[4] T.B.Gasymov, S.J.Mammadova, On convergence of spectral expansions for one discontinuous problem with spectral parameter in the boundary condition, *Trans. NAS Azerb,* 26:4 (2006), 103-116.

[5] T.B. Gasymov, A.A. Huseynli, The basis properties of eigenfunctions of a discontinuous differential operator with a spectral parameter in boundary condition, *Proceed. of IMM of NAS of Azerb.* 35 (43), 2011, 21-32.

[6] T.B. Gasymov, G.V. Maharramova, N.G. Mammadova, Spectral properties of a problem of vibrations of a loaded string in Lebesgue spaces, *Trans. of NAS of Azerb.* 38:1 (2018), 62-68.

[7] T.B. Gasymov, G.V. Maharramova, A.N. Jabrailova, Spectral properties of the problem of vibration of a loaded string in Morrey type spaces, *Proc. of IMM of NAS of Azerb.* 44:1 (2018), 116-122.

[8] B.T. Bilalov, T.B. Gasymov, G.V. Maharramova, On basicity of eigenfunctions of one discontinuous spectral problem in Morrey type spaces, *The Aligarh Bulletin of Mathematics* 35: (1-2), 2016, 119-129.

[9] T.B. Gasymov, A.M. Akhtyamov, N.R. Ahmedzade, On the basicity of eigenfunctions of a second-order differential operator with a discontinuity point in weighted Lebesgue spaces, *Proc. of IMM of NAS of Azerbaijan* 46:1 (2020), 32-44.

[10] B.T. Bilalov, T.B. Gasymov, G.V. Maharramova, Basis property of eigenfunctions in lebesgue spaces for a spectral problem with a point of discontinuity, *Differential Equations* 55:12 (2019), 1544-1553.

[11] A.Q.Akhmedov, I.Q.Feyzullayev. On completeness of Eigenfunctions of the Spectral Problem, *Caspian Journal of Applied Mathematics, Ecology and Economics* V. 10, no 2, 2022, December, pp.43-54.

[12] T.B. Gasymov, A.Q. Akhmedov , On basicity of eigenfunctions of a spectral problem in $L_p \bigoplus C$ and L_p spaces, *Baku State University Journal of Mathematics & Computer Sciences* 2024, v. 1 (1), p. 37-51.

[13] T.B. Gasymov, A.Q. Akhmedov, R.J.Taghiyeva, On Basicity of Eigenfunctions of One Spectral Problem with the Discontinuity Point in Morrey-Lebesgue Spaces, *Caspian*

Journal of Applied Mathematics, Ecology and Economics V. 12, № 2, 2023, December, pp.42-52.

[14] M.A. Naimark, Linear Differential Operators, 2nd ed. Ungar, New York, 1967.

A.Q.Ahmadov
Karabakh University, Khankendi, Azerbaijan
Baku State University, Baku, Azerbaijan
E-mail:alirza.ahmadov@karabakh.edu.az

Received 04 September 2024
Accepted 29 November 2024