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# Nodal solutions of nondifferentiable perturbations of some fourth-order half-linear boundary value problem

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**Abstract.** In this paper, we consider nondifferentiable perturbations of a certain half-linear boundary value problem for ordinary differential equations of the fourth order. Using the results of global bifurcation for the corresponding nonlinear half-eigenvalue problems, we show the existence of nodal solutions of the considered problem.

**Key Words and Phrases**: nondifferentiable perturbation, half-linear problem, half-eigenvalue, global bifurcation, nodal solution

**2010 Mathematics Subject Classifications**: 34A30, 34B05, 34B24, 34C23, 34L20, 34l30, 34K18, 47J10, 47J15

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## 1. Introduction

Consider the following nonlinear boundary value problem

$$\ell y \equiv (p\,(x)\,y'')'' - (q(x)y')' + r(x)y = \chi\tau(x)\,h(y) + \alpha(x)y^+ + \beta(x)y^-,\ x \in (0,l), \tag{1.1}$$

$$y(0) = y'(0) = y(l) = y'(l) = 0, \tag{1.2}$$

where $p\,(x)$ is a positive twice continuously differentiable function on $[0,l]$, $q(x)$ is a nonnegative continuously differentiable function on $[0,l]$, $r(x)$ is a real-valued continuous function on $[0,l]$, $\tau(x)$ is a positive continuous function on $[0,l]$, $\alpha(x)$ and $\beta(x)$ are real-valued continuous functions on $[0,l]$ such that $\alpha(x) \not\equiv -\beta(x)$. The functions $h$ has the form $h = f + g$, where the real-valued functions $f$ and $g$ are continuous on $\mathbb{R}$ and satisfy the following conditions: there exists a positive constant $M$ such that

$$\frac{|f(s)|}{|s|} \le M,\ \ s \in \mathbb{R},\ \ s \neq 0; \tag{1.3}$$

there exists positive constants $g_0$ and $g_\infty$ such that

$$\lim_{|s|\to 0\,+} \frac{g(s)}{s} = g_0 \ \text{ and } \ \lim_{|s|\to +\infty} \frac{g(s)}{s} = g_\infty. \tag{1.4}$$

 

Nonlinear boundary value problems for ordinary differential equations of fourth order arise in the mathematical modeling of various processes in mechanics, physics, and other areas of natural science. Note that problem (1.1), (1.2) describes small bending vibrations of an inhomogeneous beam, in the cross sections of which a longitudinal force acts and both ends of which are rigidly fixed (see, e.g., [15]).

Note that in the papers of many authors the existence of nodal solutions to nonlinear boundary value problems for ordinary differential equations of the second and fourth orders was investigated (see [2, 3, 5-12, 14, 16] and references therein). Using various methods, they established the conditions under which exist solutions with a fixed oscillation count of the nonlinear problems under consideration. Should be noted that in [4, 6, 14] established the existence of nodal solutions of nonlinear perturbations of half-linear boundary value problems.

In this paper, we consider the question of the existence of nodal solutions to problem (1.1), (1.2), depending on the parameter $\chi$. Under some additional conditions on the data of this problem, using the bifurcation technique, we establish intervals of this parameter in which there are solutions to problem (1.1), (1.2), contained in classes of functions with a fixed number of simple nodal zeros.

## 2. Preliminary

Let $(b.c.)$ be the set of functions $y \in C^1[0,l]$ satisfying the boundary conditions (2).

By $E$ we denote the Banach space $C^3[0,l] \cup (b.c.)$ with the norm $||y||_3 = \sum\limits_{j=0}^{3} ||y^{(j)}||_\infty$, where $||y||_\infty = \max\limits_{x\in[0,l]} |y(x)|$.

From on $\nu$ we will denote either $+$ or $-$; $-\nu$ we will denote the opposite sign to $\nu$.

For each $k \in \mathbb{N}$ and each $\nu$ let $S_k^\nu$ be the set of functions of the space $E$ constructed in [1, § 3] using the Prüfer-type transformation. Note that these classes consist of functions having the oscillatory properties of eigenfunctions (and their derivatives) of the linear spectral problem which obtained from the half-linear problem

$$\begin{cases} \ell(y) \equiv \lambda\tau(x)y + \alpha(x)y^+ + \beta(x)y^-, x \in (0,l), \\ y \in (b.c.). \end{cases} \tag{2.1}$$

by setting $\alpha \equiv \beta \equiv 0$.

We have the following oscillation theorem for problem (2.1).

**Theorem 2.1** [6, Theorem 2.1] (see also [14, Theorem 3.3]. *There exist two unbounded sequences $\{\lambda_k^+\}_{k=1}^\infty$ and $\{\lambda_k^-\}_{k=1}^\infty$ of simple half-eigenvalues of problem (2.1) such that*

$$\lambda_1^+ < \lambda_2^+ < \ \ldots\ < \lambda_k^+ < \ \ldots \ \text{ and } \ \lambda_1^- < \lambda_2^- < \ \ldots\ < \lambda_k^- < \ \ldots\ ;$$

*the half-eigenfucntions $y_k^+$ and $y_k^-$ corresponding to the half-eigenvalues $\lambda_k^+$ and $\lambda_k^-$ lie in $S_k^+$ and $S_k^-$, respectively. Furthermore, aside from solutions on the collection of the* *half-lines* $\{(\lambda_k^+, ty_k^+) : t > 0\}$ *and* $\{(\lambda_k^-, ty_k^-) : t > 0\}$ *and trivial ones, problem (2.1) has no other solutions.*

By (1.4) for the function $g$ we have the following representations:

$$g(s) = g_0 s + s\,\xi(s) \text{ and } g(s) = g_\infty s + s\zeta(s), s \in \mathbb{R}, s \neq 0, \tag{2.2}$$

where

$$\lim_{|s|\to 0+} \xi(s) = 0 \text{ and } \lim_{|s|\to +\infty} \zeta(s) = 0. \tag{2.3}$$

**Remark 2.1.** We can extend $\xi$ to $s = 0$ by $\xi(0) = 0$, and consequently, $\xi \in C(\mathbb{R})$.

Let

$$\varphi(s) = s\xi(s) \text{ and } \phi(s) = s\zeta(s), \; s \in \mathbb{R}. \tag{2.4}$$

Then it follows from (2.3) that

$$\lim_{|s|\to 0+} \frac{\varphi(s)}{s} = 0 \text{ and } \lim_{|s|\to +\infty} \frac{\phi(s)}{s} = 0. \tag{2.5}$$

**Remark 2.2.** By Remark 2.1 we have $\varphi \in C(\mathbb{R})$ and $\varphi(0) = 0$. In other hand by (2.2) and (2.4) we get

$$\phi(s) = g_0 s - g_\infty s + \varphi(s), \; s \in \mathbb{R},$$

which implies that $\phi \in C(\mathbb{R})$ and $\phi(0) = 0$.

To establish the existence of nodal solutions to problem (1.1), (1.2) we need the following result.

**Lemma 2.1.** *The following relations hold:*

$$||\varphi(u)||_\infty = o(||u||_3) \textit{ as } ||u||_3 \to 0 \; (u \in E); \tag{2.6}$$

$$||\phi(u)||_\infty = o(||u||_3) \textit{ as } ||u||_3 \to 0 \; (u \in E); \tag{2.7}$$

$$||f(u)||_\infty \leq M||u||_\infty \text{ for any } u \in E. \tag{2.8}$$

**Proof.** We define the continuous functions

$$\tilde{\varphi} : [0, +\infty) \to [0, +\infty) \text{ and } \tilde{\phi} : [0, +\infty) \to [0, +\infty)$$

as follows:

$$\tilde{\varphi}(t) = \max_{0\leq |s| \leq t} |\varphi(s)| \text{ and } \tilde{\phi}(t) = \max_{0\leq |s| \leq t} |\phi(s)|. \tag{2.9}$$

Obviously, the functions $\tilde{\varphi}$ and $\tilde{\phi}$ are nondecreasing on the half-interval $[0, +\infty)$. Hence for any $t \in (0, +\infty)$ there exists $s^*(t) \in (-t, t)$, $s^*(t) \neq 0$, such that

$$\tilde{\varphi}(t) = \max_{0\leq |s| \leq t} |\varphi(s)| = |\varphi(s^*(t))|,$$

and consequently,

$$\frac{\tilde{\varphi}(t)}{t} = \frac{|\varphi(s^*(t))|}{|s^*(t)|} \frac{|s^*(t)|}{t} \leq \frac{|\varphi(s^*(t))|}{|s^*(t)|}. \tag{2.10}$$

Since $|s^*(t)| \leq t$, by (2.5), it follows from (2.10) that

$$\lim_{t \to 0+} \frac{\tilde{\varphi}(t)}{t} = 0. \tag{2.11}$$

By the first relation of (2.9) for any $u \in E$ we get

$$\frac{|\varphi(u)|}{||u||_3} = \frac{\tilde{\varphi}(|u|)|}{||u||_3} \leq \frac{\tilde{\varphi}(||u||_\infty)}{||u||_3} \leq \frac{\tilde{\varphi}(||u||_3)}{||u||_3},$$

whence implies that

$$\frac{||\varphi(u)||_\infty}{||u||_3} \leq \frac{\tilde{\varphi}(||u||_3)}{||u||_3}. \tag{2.12}$$

By (2.11) from (2.12) we obtain (2.6).

For any $t \in (0, +\infty)$ there exists $s^\bullet(t) \in (-t, t)$, $s^\bullet(t) \neq 0$, such that

$$\tilde{\phi}(t) = \max_{0 \leq |s| \leq t} |\phi(s)| = |\phi\,(s^\bullet(t))|.$$

Then by the second relation of (2.9) we get

$$\frac{\tilde{\phi}(t)}{t} = \frac{|\phi(s^\bullet(t))|}{t} = \frac{|\phi(s^\bullet(t))|}{|s^\bullet(t)|}\frac{|s^\bullet(t)|}{t} \leq \frac{|\phi(s^\bullet(t))|}{|s^\bullet(t)|}. \tag{2.13}$$

If $t \to +\infty$, then either

(a) $|s^\bullet(t)| \to 0$, or

(b) $|s^\bullet(t)| \to +\infty$, or

(c) there exist positive constants $\kappa_0$ and $\kappa_\infty$ such that $\kappa_0 \leq |s^\bullet(t)| \leq \kappa_\infty$.

By Remark 2.2 we have $\phi \in C(\mathbb{R})$, and consequently, there exists a positive constant $K$ such that

$$|\phi(s)| \leq K \text{ for any } s \in \mathbb{R},\ \kappa_0 \leq |s| \leq \kappa_\infty. \tag{2.14}$$

In the case (a) by Remark 2.1 it follows from (2.13) that

$$\frac{\tilde{\phi}(t)}{t} = \frac{|\phi(s^\bullet(t))|}{t} \to 0 \text{ as } t \to +\infty;$$

in the case (b) by the second relation of (2.5) from (2.13) we obtain

$$\frac{\tilde{\phi}(t)}{t} \leq \frac{|\phi(s^\bullet(t))|}{|s^\bullet(t)|} \to 0 \text{ as } t \to +\infty;$$

in the case (c) by (2.14) we get

$$\frac{\tilde{\phi}(t)}{t} = \frac{|\phi(s^\bullet(t))|}{|s^\bullet(t)|}\frac{|s^\bullet(t)|}{t} \leq \frac{K}{\kappa_0}\frac{\kappa_1}{t} \to 0 \text{ as } t \to +\infty.$$

Thus we show that
$$\frac{\tilde{\phi}(t)}{t} \to 0 \text{ as } t \to +\infty. \tag{2.15}$$

Since the function $\tilde{\phi}$ is nondecreasing on $(0, +\infty)$ for any $u \in E$, $u \neq 0$, we have the following relation
$$\frac{|\phi(u)|}{||u||_3} \le \frac{\tilde{\phi}(|u|)}{||u||_3} \le \frac{\tilde{\phi}(||u||_\infty)}{||u||_3} \le \frac{\tilde{\phi}(|_3|u||_3)}{||u||_3}.$$
From the last relation we obtain
$$\frac{||\phi(u)||_\infty}{||u||_3} \le \frac{\tilde{\phi}(|_3|u||_3)}{||u||_3},$$
whence, by relation (2.13), implies (2.7).

Finally, due to (1.3) we get inequality (2.8). The proof of this lemma is complete.

## 3. Behavior of global continua of nontrivial solutions bifurcating from zero and infinity of an auxiliary nonlinear half-eigenvalue problem

To investigate the existence of nodal solutions to problem (1.1), (1.2), we consider the following nonlinear half-eigenvalue problem
$$\begin{cases} \ell(y) = \lambda\chi g_0 \tau(x) y + \alpha(x) y^+ + \beta(x) y^- + \chi\tau(x) f(y) + \chi\tau(x)\varphi(y), \ x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{3.1}$$

**Remark 3.1.** Let $\chi \in \mathbb{R}$, $\chi \neq 0$, be fixed. Then the first relation of (2.5) shows that (3.1) is a bifurcation from zero problem. Due to relations (2.6) and (2.8) of Lemma 2.1, we can apply the results of Sections 2 and 3 of [7] to problem (3.1). Then, by Lemma 2.2 and Theorem 3.1 of [7], for each $k \in \mathbb{N}$ and each $\nu$, there exists a component $C_k^\nu$ of the set of nontrivial solutions of problem (3.1) which bifurcates from $I_k \times \{0\}$, is contained in $\mathbb{R} \times S_k^\nu$ and is unbounded in $\mathbb{R} \times E$ (in this case either $C_k^\nu$ meet $(\lambda, \infty)$ for some $\lambda \in \mathbb{R}$ or the projection of $C_k^\nu$ onto $\mathbb{R} \times \{0\}$ is unbounded), where
$$I_k^\nu = \left[\tilde{\lambda}_k^\nu - \frac{N_\alpha + N_\beta}{\chi\tilde{\tau}_0} - \frac{M}{g_0}, \tilde{\lambda}_k^\nu + \frac{N_\alpha + N_\beta}{\chi\tilde{\tau}_0} + \frac{M}{g_0}\right], \tag{3.2}$$
$\tilde{\lambda}_k^+$ and $\tilde{\lambda}_k^-$ are $k$-th half-eigenvalues of the half-linear problem
$$\begin{cases} \ell(y) = \lambda\chi g_0 \tau(x) y + \alpha(x) y^+ + \beta(x) y^-, x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{3.3}$$
$$\tilde{\tau}_0 = g_0\tau_0, \tau_0 = \min_{x \in [0,l]} \tau(x), N_\alpha = \max_{x \in [0,l]} |\alpha(x)|, N_\beta = \max_{x \in [0,l]} |\beta(x)|.$$

By (3.3) it follows from (2.1) that
$$\lambda_k^\nu = \tilde{\lambda}_k^\nu \chi g_0 \ \text{ for each } k \in \mathbb{N} \text{ and each } \nu, \tag{3.4}$$

where $\lambda_k^+$ and $\lambda_k^-$ are $k$-th half-eigenvalues of the half-linear problem (2.1). Then, by (3.4), from (3.2) we get

$$I_k^\nu = \left[\frac{\lambda_k^+}{\chi g_0} - \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} - \frac{M}{g_0}, \frac{\lambda_k^\nu}{\chi g_0} + \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} + \frac{M}{g_0}\right]. \tag{3.5}$$

**Remark 3.2.** By the second relations of (2.2) and (2.4), we rewrite problem (3.1) in the following form

$$\begin{cases} \ell(y) = \left(\lambda + \frac{g_\infty}{g_0} - 1\right) \chi g_0 \tau(x) y + \alpha(x) y^+ + \beta(x) y^- + \chi\tau(x) f(y) + \\ +\chi\tau(x)\phi(y), \ x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{3.6}$$

The second relation of (2.5) shows that problem (3.6) is a bifurcation at infinity problem. By the relations (2.6)-(2.8) of Lemma 2.1, we can apply the results of [6, Section 3] and [8, Section 3] to problem (3.6). Then, by [8, Theorem 3.1 and Theorem 3.2], for each $k \in \mathbb{N}$ and each $\nu$, there exists a component $D_k^\nu$ of the set of nontrivial solutions of problem (3.6) which emanates from $J_k \times \{\infty\}$, is contained in $\mathbb{R} \times S_k^\nu$ and either meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$ or its projection onto $\mathbb{R} \times \{0\}$ is unbounded, where

$$J_k^\nu = \left[\bar\lambda_k^\nu - \frac{N_\alpha + N_\beta}{\tilde\tau_0} - \frac{M}{g_0}, \bar\lambda_k^\nu + \frac{N_\alpha + N_\beta}{\tilde\tau_0} + \frac{M}{g_0}\right], \tag{3.7}$$

$\bar\lambda_k^+$ and $\bar\lambda_k^-$ are $k$-th half-eigenvalues of the half-linear problem

$$\begin{cases} \ell(y) = \left(\lambda + \frac{g_\infty}{g_0} - 1\right) \chi g_0 \tau(x) y + \alpha(x) y^+ + \beta(x) y^-, x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{3.8}$$

By (3.8) it follows from (2.1) that for each $k \in \mathbb{N}$ and each $\nu$ the relation

$$\lambda_k^\nu = \left(\bar\lambda_k^\nu + \frac{g_\infty}{g_0} - 1\right) \chi g_0$$

holds. Then it follows from last relation that

$$\bar\lambda_k^\nu = \frac{\lambda_k^\nu}{\chi g_0} - \frac{g_\infty}{g_0} + 1.$$

Consequently, from (3.7) we obtain

$$J_k^\nu = \left[\frac{\lambda_k^\nu}{\chi g_0} - \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} - \frac{g_\infty + M}{g_0} + 1, \frac{\lambda_k^\nu}{\chi g_0} + \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} - \frac{g_\infty - M}{g_0} + 1\right]. \tag{3.9}$$

We have the following result.

**Lemma 3.2.** *If $C_k^\nu$ meets $(\lambda, \infty)$ for some $\lambda \in \mathbb{R}$, then $\lambda \in J_k$, and if $D_k^\nu$ meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$, then $\lambda \in I_k$.*

The proof of this lemma follows from [7, Lemma 2.2] and [8, Remark 3.2] (see also [13, Theorem 3.3]) due to the above arguments.

**Lemma 3.2.** *For each $k \in \mathbb{N}$ and each $\nu$, the projection of $C_k^\nu$ and $D_k^\nu$ onto $\mathbb{R} \times \{0\}$ are bounded.*

**Proof.** Let $k \in \mathbb{N}$ and $\nu$ are arbitrary fixed and let $(\hat{\lambda}, \hat{y}) \in \mathbb{R} \times S_k^\nu$ be the solution of problem (2.15), where $|\lambda|$ is large enough.

We introduce the following notations:

$$\hat{\psi}(x) = \begin{cases} \frac{f(\hat{y}(x))}{\hat{y}(x)} & \text{if } \hat{y}(x) \neq 0, \\ 0 & \text{if } \hat{y}(x) = 0, \end{cases}, \text{ and } \hat{\xi}(x) = \xi(\hat{y}(x)), x \in [0, l]. \tag{3.10}$$

Then $\lambda = \hat{\lambda}_k^\nu$, where $\hat{\lambda}_k^\nu$ is the $k$-th half-eigenvalue of the half-linear problem

$$\begin{cases} \ell(y) = \lambda \chi g_0 \tau(x) y + a(x) y^+ + b(x) y^- + \chi \tau(x)(\hat{\psi}(x) + \hat{\zeta}(x)) y, \ x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{3.11}$$

In view of condition (1.3) by the first notation of (3.10) we obtain

$$|\hat{\psi}(x)| \leq M, \ x \in [0, l]. \tag{3.12}$$

It follows from relations (2.2) and (2.4) that

$$\xi(s) = g_\infty - g_0 + \zeta(s), \ s \in \mathbb{R}.$$

Since $\zeta \in C(\mathbb{R})$ by (2.3) there exists a positive constant $L$ such that

$$|\zeta(s)| \leq L|s|, \ s \in \mathbb{R}.$$

which, by the second relation of (3.10), implies that

$$|\hat{\xi}(x)| \leq L, \ x \in [0, l]. \tag{3.13}$$

Then, in view of (3.12) and (3.13), it follows from [3, relation (2.16)] that

$$|\hat{\lambda}_k^\nu - \lambda_k^\nu| \leq \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} + \frac{M + L}{g_0}. \tag{3.14}$$

Therefore, we have the following estimate

$$|\hat{\lambda}| = |\hat{\lambda}_k^\nu| \leq |\hat{\lambda}_k^\nu - \lambda_k^\nu| + |\lambda_k^\nu| \leq |\lambda_k^\nu| + \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} + \frac{M + L}{g_0},$$

which contradicts the fact that $|\hat{\lambda}|$ is large enough.

Thus, we have shown that the projection of $C_k^\nu$ onto $\mathbb{R} \times \{0\}$ is bounded. In a similar way it can be shown that the projection of $D_k^\nu$ onto $\mathbb{R} \times \{0\}$ is also bounded. The proof of this lemma is complete.

**Corollary 3.1.** *For each $k \in \mathbb{N}$ and each $\nu$ the components $C_k^\nu$ and $D_k^\nu$ of the set of nontrivial solutions of problem (3.1) coincide.*

Thus, by Corollary 3.1, we have the following result.

**Theorem 3.1.** *For each $k \in \mathbb{N}$ and each $\nu$ the component $C_k^\nu$ of the set of nontrivial solutions to problem (3.1) is contained in $\mathbb{R} \times S_k^\nu$ and meets the intervals $I_k \times \{0\}$ and $J_k \times \{\infty\}$.*

## 4. Existence of nodal solutions to problem (1.1), (1.2)

The following theorem is the main result of this paper.

**Theorem 4.1.** *Let the following conditions hold:*

*(i)* $g_0 > M$ *and* $g_\infty > M$*;*

*(ii) for some* $k \in \mathbb{N}$ *and some* $\nu$, $\lambda_k^\nu - \frac{N_a+N_b}{\tau_0} > 0$, *and either*

$$\frac{\lambda_k^\nu}{g_0 - M} + \frac{N_\alpha + N_\beta}{\tau_0(g_0 - M)} < \chi < \frac{\lambda_k^\nu}{g_\infty + M} - \frac{N_\alpha + N_\beta}{\tau_0(g_\infty + M)}\,, \tag{3.15}$$

*or*

$$\frac{\lambda_k^\nu}{g_\infty - M} + \frac{N_\alpha + N_\beta}{\tau_0(g_\infty - M)} < \chi < \frac{\lambda_k^\nu}{g_0 + M} - \frac{N_\alpha + N_\beta}{\tau_0(g_0 + M)}\,. \tag{3.16}$$

*Then there exists a solution* $\upsilon_k^\nu$ *of problem (1.1), (1.2) such that* $\upsilon_k^\nu \in S_k^\nu$, *i.e. the function* $\upsilon_k^\nu$ *has exactly* $k-1$ *simple nodal zeros in the interval* $(0, l)$.

**Proof.** It is obvious that any nontrivial solution $(\lambda, y) \in \mathbb{R} \times E$ with $\lambda = 1$ of problem (3.1) is a nontrivial solution of problem (1.1), (1.2). Then, according to Theorem 3.1, if for some $k \in \mathbb{N}$ the right end of the interval $I_k$ is to the left of 1 and the left end of the interval $J_k$ is to the right of 1 on the real axis, or the right end of the interval $J_k$ is to the left of 1 and the left end of the interval $I_k$ is to the right of 1 on the real axis, then problem (1.1), (1.2) will have a solution that is contained in the class $S_k^\nu$.

Let conditions (i) and (ii) of this theorem be satisfied. If (3.15) holds, then we have the following relations

$$\frac{\lambda_k^\nu}{g_0 - M} + \frac{N_\alpha + N_\beta}{\tau_0(g_0 - M)} < \chi \ \text{ and } \ \chi < \frac{\lambda_k^\nu}{g_0 + M} - \frac{N_\alpha + N_\beta}{\tau_0(g_0 + M)}\,,$$

which implies that

$$\frac{\lambda_k^\nu}{\chi g_0} + \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} + \frac{M}{g_0} < 1 \ \text{ and } \ 0 < \frac{\lambda_k^\nu}{\chi g_0} - \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} - \frac{g_\infty + M}{g_0}. \tag{3.17}$$

From (3.17) we obtain

$$\frac{\lambda_k^\nu}{\chi g_0} + \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} + \frac{M}{g_0} < 1 < \frac{\lambda_k^\nu}{\chi g_0} - \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} - \frac{g_\infty + M}{g_0} + 1,$$

which show that the right end of the interval $I_k$ is to the left of 1, and the left end of the interval $J_k$ is to the right of 1 on the real axis.

If (3.16) is satisfied, then it can be shown in a similar way that the right end of the interval $J_k$ is to the left of 1, and the left end of the interval $I_k$ is to the right of 1 on the real axis. The proof of this theorem is complete.

## References

[1] Z.S. Aliev, On the global bifurcation of solutions of some nonlinear eigenvalue problems for ordinary differential equations of fourth order, *Sb. Math.*, **207**(12) (2016), 1625-1649.

[2] Z.S. Aliyev, Y.N. Aliyeva, Existence of nodal solutions to some nonlinear boundary value problems for ordinary differential equations of fourth order, *Electron. J. Qual. Theory Differ. Equ.* (25) (2024), 1-13.

[3] Z.S. Aliyev, K.R. Rahimova, Existence of nodal solutions of some nonlinear Sturm-Liouville problem with a spectral parameter in the boundary condition, *Azerb. J. Math.* **15**(1) (2025), 243-256.

[4] H. Berestycki, On some nonlinear Sturm-Liouville problems, *J. Differential Equations*, **26**(3) (1977), 375-390.

[5] Y. Cui, J. Sun, Y. Zou, Global bifurcation and multiple results for Sturm- Liouville problems, *J. Comput. Appl. Math.* **235**(8) (2011), 2185-2192.

[6] M.M. Mammadova, On some asymptotically half-linear eigenvalue problem for ordinary differential equations of fourth order, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **48**(1) (2022), 113-122.

[7] M.M. Mammadova, Global bifurcation from zero in nondifferentiable perturbations of half-linear fourth-order eigenvalue problems, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **49**(1) (2023), 28-37.

[8] M.M. Mammadova, Global bifurcation from infinity of nondifferentiable perturbations of half-linear eigenvalue problems for ordinary differential equations of fourth order, *Casp. J. Appl. Math., Ecol. Econ.* **11**(2), (2023), 15-21.

[9] R. Ma and G. Dai, Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity, *J. Functional Analysis* **265**(8) (2013), 1443-1459.

[10] R. Ma, B. Thompson, Nodal solutions for nonlinear eigenvalue problems. *Nonlinear Anal.* **59**(5) (2005), 707-718.

[11] R. Ma, B. Thompson, Multiplicity results for second-order two-point boundary value problems with nonlinearities across several eigenvalues, *Appl. Math. Lett.* **18**(5) (2005), 587-595.

[12] Naito Y, Tanaka S. On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations, *Nonlinear Anal.* 56(4) (2004), 919-935.

[13] B.P. Rynne, Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable, *J. Math. Anal. Appl.*, **228**(1) (1998), 141-156.

[14] B.P. Rynne, Half-eigenvalues of self adjoint, $2m$th order differential operators and semilinear problems with jumping nonlinearities, *Differential Integral Equations* 14(9) (2001), 1129-1152.

[15] M. Roseau, *Vibrations in mechanical systems. Analytical methods and applications*, Springer-Verlag, Berlin 1987.

[16] Tanaka S. On the uniqueness of solutions with prescribed numbers of zeros for a two-point boundary value problem, *Differential Integral Equations* 20(1) (2007), 3-104.

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